Math 1231 Fall 2022
Single-Variable Calculus I Section 13
Mastery Quiz 2
Due Thursday, September 15

This week's mastery quiz has three topics. Everyone should submit M1 and S2. If you already have a 2/2 on S1, you don't need to submit it again.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Thursday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Secondary Topic 1: Estimation
- Secondary Topic 2: The Squeeze Theorem

Name:

Recitation Section:

Major Topic 1: Computing Limits

(a) $\lim_{x\to -2} \dfrac{x+2}{\sqrt{x+6}-2} =$

Solution:

$$\lim_{x\to -2} \frac{x+2}{\sqrt{x+6}-2} = \lim_{x\to -2} \frac{(x+2)(\sqrt{x+6}+2)}{x+6-4}$$
$$= \lim_{x\to -2} \frac{\sqrt{x+6}+2}{1} = 4.$$

(b) Compute $\lim_{x\to 1} \dfrac{\sin(3x-3)\sin(x-1)}{(x-1)^2} =$

Solution:

$$\lim_{x\to 1} \frac{\sin(3x-3)\sin(x-1)}{(x-1)^2} = \lim_{x\to 1} \frac{\sin(3x-3)}{x-1}\frac{\sin(x-1)}{x-1}$$
$$= \lim_{x\to 1} 3\frac{\sin(3x-3)}{3x-3} = 3.$$

(c) $\lim_{x\to -1} \dfrac{1-x}{1+x} =$

Solution: The limit of the top is 2 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the denominator can be positive or negative, we can't be more specific.

Secondary Topic 1: Estimation

(a) Suppose $f(x) = \sqrt{x-3}$, and we want an output of approximately 2. What input a should we aim for? Find a δ so that if our input is $a \pm \delta$ then our output will be $2 \pm .5$. Explain how you found this δ and why it should give us what we want.

Solution: We want an input of about $a = 7$. By solving the equation we can see that if

$$\begin{aligned} f(x) &= 1.5 \\ x-3 &= 1.5^2 = 2.25 \\ x &= 5.25 \end{aligned} \qquad \begin{aligned} f(x) &= 2.5 \\ x-3 &= 2.5^2 = 6.25 \\ x &= 9.25 \end{aligned}$$

so we want x in $(5.25, 9.25)$. This ranges from $7-1.75$ to $7+2.25$, so we take $\delta = 1.75$ as the smaller of these two distances.

(b) We can make a pound of charcoal from nine pounds of wood, and we want to produce about 8 pounds of charcoal. Find a formula for δ in terms of ϵ, so that if the error in the amount of *wood* is less than δ then the error in the amount of charcoal is less than ϵ. Make sure your formula gives the **largest δ possible**, and justify your answer.

Solution: The amount of charcoal we produce is $W/9$, so our output error is $|W/9 - 8| = \frac{1}{9}|W - 72|$, which we want to be less than ε. So we get

$$|W/9 - 8| = \frac{1}{9}|W - 72| < \varepsilon$$
$$|W - 72| < 9\varepsilon.$$

So if we take $\delta = 9\varepsilon$, then whenever the error in the amount of wood we use, $|W - 72|$, is less than δ, then the error in our amount of charcoal should be less than ε.

Secondary Topic 2: The Squeeze Theorem

Show that $\lim_{x\to 2}(x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right) = 0$.

Solution: We know that

$$-1 \le \sin\left(\frac{1}{x-2}\right) \le 1$$
$$0 \le 1 + \sin\left(\frac{1}{x-2}\right) \le 2$$
$$0 \left| \le 1 + \sin\left(\frac{1}{x-2}\right)\right| \le 2$$
$$0 \le \left|(x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right)\right| \le 2|x-2|.$$

Since $\lim_{x\to 2} 0 = 0$ and $\lim_{x\to 2} 2|x-2| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 2}(x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right) = 0$.