Math 1231: Single-Variable Calculus 1
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Recitation 2

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Problem 1. Let $f(x) = 5x + 2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x\to 2} f(x)$.

(a) If x is about 2, what should $f(x)$ be?

(b) Write down expressions using absolute value for the input and output errors.

(c) If we want $\varepsilon = 1$, what does δ need to be?

(d) Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

(e) Try to write a full proof.

Solution:

(a) $f(x) \approx 12$.

(b) Output error is $|f(x) - 12|$ or $|5x + 2 - 12|$, which we can simplify to $|5x - 10|$. Input error is $|x - 2|$.

(c) We want $|5x - 10| < 1$, and dividing by 5 gives $|x - 2| < 1/5$. So we'd need $\delta = 1/5$.

(d) We want $|5x - 10| < \varepsilon$, and dividing by 5 gives $|x - 2| < \varepsilon/5$. So we'd need $\delta = \varepsilon/5$.

(e) Let $\varepsilon > 0$ and set $\delta = \varepsilon/5$. Then if $0 < |x - 2| < \delta = \varepsilon/5$ we compute that

$$|f(x) - 12| = |5x - 10| = 5|x - 2| < 5 \cdot \varepsilon/5 = \varepsilon.$$

Problem 2 (Optional)**.** Let $g(x) = x^2$. We want to use an $\varepsilon - \delta$ argument to compute $\lim_{x \to 0} g(x)$.

(a) If x is about 0, what should $g(x)$ be?

(b) Write down expressions using absolute value for the input and output errors.

(c) If we want $\varepsilon = 1$, what does δ need to be? What about $\varepsilon = 1/4$?

(d) Find a formula for δ in terms of ε (same form as $\delta = \varepsilon/3$ or $\delta = \varepsilon$).

(e) Try to write a full proof.

Solution:

(a) $g(x) \approx 0$.

(b) Output error is $|g(x) - 0|$ or $|x^2 - 0|$, which we can simplify to x^2. Input error is $|x - 0|$, which we can simplify to $|x|$.

(c) We want $x^2 < 1$, and taking square roots gives $|x| < 1$, so we need $\delta = 1$.

If we want $x^2 < 1/4$ then taking square roots gives $|x| < 1/2$, so we need $\delta = 1/2$.

Note that in both cases the absolute value matters; the square root of x^2 is always positive, and thus equals $|x|$.

(d) We want $x^2 < \varepsilon$, and taking square roots gives $|x| < \sqrt{\varepsilon}$. So we take $\delta = \sqrt{\varepsilon}$.

(e) Let $\varepsilon > 0$ and set $\delta = \sqrt{\varepsilon}$. Then if $0 < |x| < \delta = \sqrt{\varepsilon}$ we compute that

$$|g(x) - 0| = x^2 = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon.$$

Now let's look at easier ways to actually compute limits.

Problem 3. Let $f(x) = \frac{x-1}{x^2-1}$.

(a) What is $f(2)$? Is f continuous at 2?

(b) What is $\lim_{x \to 2} f(x)$?

(c) What is $f(1)$? Is f continuous at 1?

(d) What function can we find that's almost the same as f, but defined and continuous at 1? (Is this function the same as f?)

(e) What is $\lim_{x \to 1} f(x)$?

Solution:

(a) $f(2) = 1/3$, and f is continuous here since it's a reasonable functions.

(b) $\lim_{x\to 2} f(x) = 1/3$.

(c) $f(1)$ isn't defined, and thus f is not continuous at 1.

(d) $\frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)}$ is almost the same as $\frac{1}{x+1}$.

(e) $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2}$.

Note we're using the Almost Identical Functions principle here. The function f was *not* continuous at 1, because it's not defined there. But we can replace it by an almost identical function that is continuous, and then that limit is simple to compute.

Problem 4 (Optional)**.** Let $g(x) = \frac{(x+1)^2-1}{x+2}$.

(a) Is g continuous where it's defined? Where is it undefined?

(b) Can you find a function that's almost identical to g but continuous everywhere?

(c) What is $\lim_{x\to -2} g(x)$?

Solution:

(a) g is a reasonable function so it's continuous where it's defined, but it isn't defined at $x = -2$.

(b) $\frac{x^2+2x+1-1}{x+1} = \frac{x(x+2)}{x+2}$ is almost the same as x. So $g(x)$ is almost the same as x.

(c) $\lim_{x\to -2} g(x) = \lim_{x\to -2} x = -2$.

Note that $\frac{x(x+2)}{x+2} \neq x$, but their limits at 0 are the same because the functions are the same near 0 (and in fact everywhere except at 0).

Problem 5. Let $h(x) = \frac{x-1}{\sqrt{5-x}-2}$.

(a) Is this function continuous where it's defined? Where is it undefined?

(b) We can factor an $x-1$ out of the top, but we can't obviously factor one out of the bottom. We need to use an algebraic trick make the $x-1$ appear. What tricks do we have that might work?

(c) What is $\lim_{X\to 1} h(x)$?

Solution:

(a) The function is reasonable, so it's continuous where defined. IT's undefined at $x = 1$ and also at $x > 5$.

(b) Here we need to multiply by the conjugate. We can compute

$$\begin{aligned}\frac{x-1}{\sqrt{5-x}-2} &= \frac{x-1}{\sqrt{5-x}-2}\frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)}.\end{aligned}$$

This function is not the same as $-\sqrt{5-x}-2$, but it's very close.

(c)

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{\sqrt{5-x}-2} &= \lim_{x\to 1}\frac{x-1}{\sqrt{5-x}-2}\frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\ &= \lim_{x\to 1}\frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\ &= \lim_{x\to 1}\frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)} \\ &= \lim_{x\to 1} -(\sqrt{5-x}+2) = -4.\end{aligned}$$

Problem 6. Let

$$f(x) = \begin{cases} x^2+1 & x > 2 \\ 9-2x & x < 2 \end{cases}$$

Can we extend this to a continuous function on all reals?

(a) Where is f continuous? Where is it discontinuous?

(b) What value "should" $f(x)$ have for $x = 2$?

(c) Can you define a function that's Almost Identical to $f(x)$, but continuous at all reals?

Solution:

(a) f is continuous everywhere it's given by algebra, so everywhere except at $x = 2$.

(b) We can compute $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 9-2x = 5$, and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2+1 = 5$, so the limit at 2 exists and we have $\lim_{x\to 2} f(x) = 5$.

(c)

$$g(x) = \begin{cases} x^2+1 & x \geq 2 \\ 9-2x & x \leq 2 \end{cases}$$