Math 1231: Single-Variable Calculus 1
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Recitation 3

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Problem 1. Let $h(x) = \frac{x-1}{\sqrt{5-x}-2}$.

(a) Is this function continuous where it's defined? Where is it undefined?

(b) We can factor an $x-1$ out of the top, but we can't obviously factor one out of the bottom. We need to use an algebraic trick make the $x-1$ appear. What tricks do we have that might work?

(c) What is $\lim_{X \to 1} h(x)$?

Solution:

(a) The function is reasonable, so it's continuous where defined. IT's undefined at $x = 1$ and also at $x > 5$.

(b) Here we need to multiply by the conjugate. We can compute

$$\begin{aligned} \frac{x-1}{\sqrt{5-x}-2} &= \frac{x-1}{\sqrt{5-x}-2} \frac{\sqrt{5-x}+2}{\sqrt{5-x}+2} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4} \\ &= \frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)}. \end{aligned}$$

This function is not the same as $-\sqrt{5-x}-2$, but it's very close.

(c)

$$\begin{aligned}
\lim_{x\to 1}\frac{x-1}{\sqrt{5-x}-2} &= \lim_{x\to 1}\frac{x-1}{\sqrt{5-x}-2}\frac{\sqrt{5-x}+2}{\sqrt{5-x}+2}\\
&= \lim_{x\to 1}\frac{(x-1)(\sqrt{5-x}+2)}{(5-x)-4}\\
&= \lim_{x\to 1}\frac{(x-1)(\sqrt{5-x}+2)}{-(x-1)}\\
&= \lim_{x\to 1}-(\sqrt{5-x}+2) = -4.
\end{aligned}$$

Problem 2. We want to compute $\lim_{x\to 3}(x-3)\left(5\sin\left(\frac{1}{x-3}\right)-2\right)$.

(a) Is this function continuous where defined? Is it defined at 3?

(b) Does this look like the squeeze theorem might be helpful? Why?

(c) Can you find bounds for the most difficult piece of the function? That is, find a number that's always greater and a number that's always less.

(d) Can we use that to find bounds for $5\sin\left(\frac{1}{x-3}\right)-2$?

If you haven't already, check in with your neighbors and make sure everyone has the same (or similar) bounds. The next part won't make sense if you have something wrong here.

(e) We want to multiply by $x-3$, but that doesn't work because it might be negative, so now we want to put our work inside absolute value signs. Can we just take the absolute value of all three sides of the inequality you got in (d)?

(f) Can we find upper and lower bounds for $\left|5\sin\left(\frac{1}{x-3}\right)-2\right|$? What has to be true here?

(g) Use those bounds, and the squeeze theorem, to show that

$$\lim_{x\to 3}(x-3)\left(5\sin\left(\frac{1}{x-3}\right)-2\right)=0.$$

Solution:

(a) It's continuous where defined, but it isn't defined at 3, because you'd have to divide by 0.

(b) We're looking for a term going to zero multiplied by a term that is messy. We have both of them.

(c) We know that $-1 \leq \sin\left(\frac{1}{x-3}\right) \leq$.

(d)

$$\begin{aligned} -1 &\leq \sin\left(\frac{1}{x-3}\right) \leq 1 \\ -5 &\leq 5\sin\left(\frac{1}{x-3}\right) \leq 5 \\ -7 &\leq 5\sin\left(\frac{1}{x-3}\right) - 2 \leq 3. \end{aligned}$$

(e) We can't do this, because we'd get

$$7 \leq \left|5\sin\left(\frac{1}{x-3}\right) - 2\right| \leq 3.$$

Obviously 7 is not less than 3.

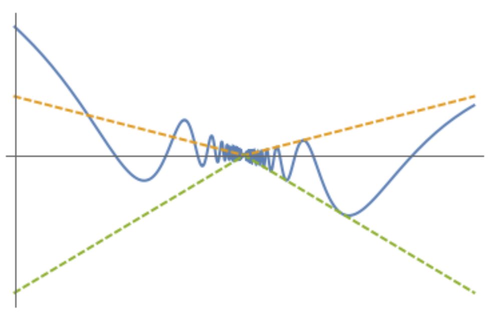
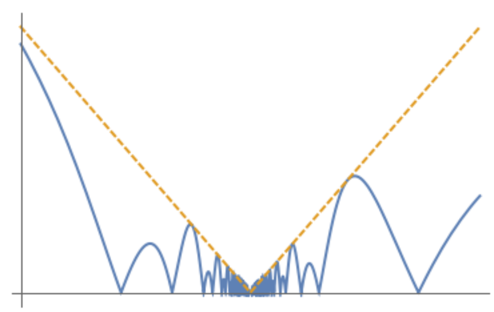
(f) The most obvious answer, which is wrong, is

$$\left|5\sin\left(\frac{1}{x-3}\right) - 2\right| \leq 3.$$

But this *isn't true*, because the complicated bit can be -7, and $|-7| = 7 > 3$. The largest *magnitude* we can get for the middle bit is $|-7| = 7$, so we want

$$\left|5\sin\left(\frac{1}{x-3}\right) - 2\right| \leq 7.$$

Compare the graphs below:

(g) Now we can multiply through by $|x-3|$, and get

$$0 \leq \left|(x-3)\left(5\sin\left(\frac{1}{x-3}\right) - 2\right)\right| \leq |7(x-3)|.$$

Then we can compute that $\lim_{x\to 3} 0 = 0$ and $\lim_{x\to 3} |7(x-3)| = 0$, so by the squeeze theorem we know that $\lim_{x\to 3}(x-3)\left(5\sin\left(\frac{1}{x-3}\right)\right) = 0$.

Problem 3. We want to compute $\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3}$.

(a) What rule do we know we need to invoke here?

(b) What θ are we going to need to pick for this to work out, and why?

(c) Do algebra so that you can invoke the small angle approximation. What is the limit? (Are you using the AIF property?)

(d) Go back to the beginning, and see what our heuristic idea that $\sin(\theta) \approx \theta$ would have told you. Does that match with what you got?

Solution:

(a) We need to use the small angle approximation, because this problem requires us to make trig interact with algebra.

(b) We basically have to take $\theta = x^2 - 9$, because that's what's inside the sin.

(c) We need to get a θ on the bottom as well. So we take

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} =^{AIF} \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9}$$
$$= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.$$

You have to use AIF in that first equality, because you're making the function undefined at -3.

There are a couple different ways to make this algebra work out. You might also try

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} =^{AIF} \lim_{x\to 3} \frac{\sin(x^2-9)/x^2-9 \cdot x^2-9}{x-3}$$
$$= \lim_{x\to 3} \frac{x^2-9}{x-3} =^{AIF} \lim_{x\to 3} x+3 = 6,$$

which is essentially the same logic.

(d)

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} = \lim_{x\to 3} \frac{\sin(x^2-9)}{x-3} \cdot \frac{x+3}{x+3} = \lim_{x\to 3} \frac{\sin(x^2-9)(x+3)}{x^2-9}$$
$$= \lim_{x\to 3} \frac{\sin(x^2-9)}{x^2-9} \cdot \lim_{x\to 3} x+3 = 1 \cdot (3+3) = 6.$$

Problem 4. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

(a) To start: what is $\lim_{x\to 0} 1/x$, and why?

(b) Let's look at $\lim_{x\to 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?

(c) How do we actually compute $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?

(d) Now consider $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?

(e) Compute $\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?

(f) Now consider $\lim_{x\to 0} 1/x + 1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?

(g) Finally, let's look at $\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?

(h) What pattern do you see from all of these?

Solution:

(a) $\lim_{x\to 0} \frac{1^{\nearrow 1}}{x_{\searrow 0}} = \pm\infty$.

(b) This looks like $\infty + \infty$ as an indeterminate form.

(c) We see $\lim_{x\to 0} \frac{1}{x} + \frac{1}{x} = \lim_{x\to 0} \frac{2^{\nearrow 2}}{x_{\searrow 0}} = \pm\infty$. This seems to make sense; $\infty + \infty = \infty$ is perfectly reasonable.

(d) We already know $\lim_{x\to 0} \frac{1}{x} = \pm\infty$. We can compute that

$$\lim_{x\to 0} \frac{x-1^{\nearrow -1}}{x-x^2{}_{\searrow 0}} = \pm\infty.$$

So this is again $\infty + \infty$.

(e) By combining fractions, we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{x-1}{x-x^2} = \lim_{x\to 0} \frac{1-x}{x-x^2} + \frac{x-1}{x-x^2} = \lim_{x\to 0} 0 = 0.$$

So here $\infty + \infty = 0$.

(f) We have a $\pm\infty$ plus a $+\infty$, so we get $\infty + \infty$ again. When we combine them into one term we get

$$\lim_{x\to 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x\to 0} \frac{x+1^{\nearrow 1}}{x^2{}_{\searrow 0}} = +\infty$$

since the denominator is $x^2 \geq 0$. So here $\infty + \infty = +\infty$.

We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ "faster" than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

(g) We compute $\lim_{x\to 0} \frac{x-3x+2^{\nearrow 2}}{x^2-2x{}_{\searrow 0}} = \pm\infty$, so this is, again, $\infty + \infty$. The actual limit is

$$\lim_{x\to 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x-2+x^2-3x+2}{x^2-2x} = \lim_{x\to 0} \frac{x^2-2x}{x^2-2x} = \lim_{x\to 0} 1 = 1.$$

So here $\infty + \infty = 1$.

(h) In conclusion, if you know something looks like $\infty+\infty$, you don't really know anything about it at all.