Math 2233 Practice Midterm 2 Solutions

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Problem 1 (M3). (a) Find and classify the critical points of $f(x, y) = 2x^3 + 6xy + 3y^2$.

Solution: We have

$$f_x(x, y) = 6x^2 + 6y$$
$$f_y(x, y) = 6x + 6y$$

This gives us $y = -x$, and thus we have $x^2 - x = 0$ so x is either 0 or 1. Thus our critical points are $(0, 0)$ and $(1, -1)$.

We have

$$\begin{aligned} f_{xx}(x, y) &= 12x & f_{xx}(0, 0) &= 0 & f_{xx}(1, -1) &= 12 \\ f_{xy}(x, y) &= 6 & f_{xy}(0, 0) &= 6 & f_{xy}(1, -1) &= 6 \\ f_{yy}(x, y) &= 6 & f_{yy}(0, 0) &= 6 & f_{yy}(1, -1) &= 6. \end{aligned}$$

Then for $(0, 0)$ we have $D = 0 \cdot 6 - 6^2 = -36 < 0$, so we have a saddle point.

For $(1, -1)$ we have $D = 12 \cdot 6 - 6^2 = 36 > 0$, and $f_{xx}(1, -1) = 12 > 0$. So this is a local minimum.

(b) Find the maximum and minimum values of $f(x, y) = 20 - 4x^2 - y^2$ on the disk $x^2 + y^2 \leq 4$.

Solution: For interior critical points, we have $\nabla f(x, y) = (-8x, -2y)$, which gives the equations $-8x = 0$ and $-2y = 0$. Thus the only interior critical point is $(0, 0)$, and we compute $f(0, 0) = 20$.

On the boundary, we have

$$-8x = 2\lambda x$$
$$-2y = 2\lambda y.$$

The second condition gives that $y = 0$ or $\lambda = -1$. If $y = 0$ then $x = \pm 2$; if $\lambda = -1$ then the first equation tells us that $x = 0$ and thus $y = \pm 2$. So we have four critical points:

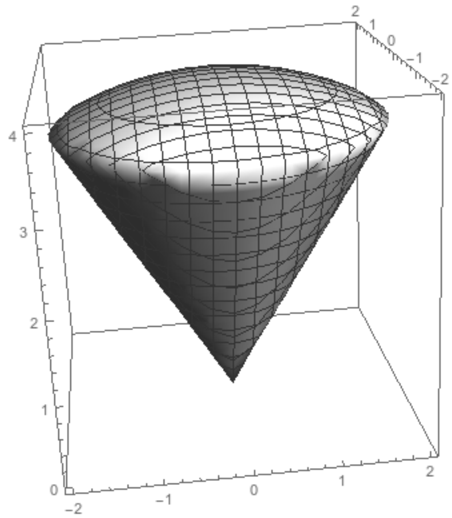
$$\begin{aligned} f(2, 0) &= 4 \\ f(-2, 0) &= 4 \\ f(0, 2) &= 16 \\ f(0, -2) &= 16. \end{aligned}$$

So the absolute maximum on the disk is 20, and the absolute minimum is 4.

Problem 2 (M4). Let R be the spherical wedge bounded by a sphere of radius 4 centered at the origin, and the cone given by $z = \sqrt{3x^2 + 3y^2}$ (as shown below). Let $f(x, y, z) = z$.

(a) Set up integrals to compute $\int_R f \, dV$ in cartesian, cylindrical, and spherical coordinates.

(b) Choose one of these integrals and evaluate it.



Solution:

(a) We see that these intersect at the circle $x^2 + y^2 + 3x^2 + 3y^2 = 16$, or in other words $x^2 + y^2 = 4$, so the circle of radius 2 at the level $z = \sqrt{12} = 2\sqrt{3}$.

If we draw a triangle from the side, we see that we have a triangle with opposite side of length 2 and hypotenuse of length 4, so $\sin\phi = 1/2$. Thus $\phi = \pi/6$.

$$\begin{aligned} I &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} z \, dz \, dy \, dx \\ &= \int_0^{2\pi} \int_0^2 \int_{r\sqrt{3}}^{\sqrt{16-r^2}} zr \, dz \, dr \, d\theta \\ &= \int_0^4 \int_0^{2\pi} \int_0^{\pi/6} \rho \cos\phi \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho \end{aligned}$$

(b) I really hope everyone picks the spherical integral. We compute

$$\begin{aligned} I &= \int_0^4 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos\phi \sin\phi \, d\phi \, d\theta \, d\rho \\ &= \frac{1}{2} \int_0^4 \int_0^{2\pi} \rho^3 \sin^2\phi |_0^{\pi/6} \, d\theta \, d\rho \\ &= \frac{1}{8} \int_0^4 \int_0^{2\pi} \rho^3 \, d\theta \, d\rho \\ &= \frac{\pi}{4} \int_0^4 \rho^3 \, d\rho = \frac{\pi}{4} \frac{\rho^4}{4} |_0^4 = 16\pi. \end{aligned}$$

(c) Compute the integral of the function $f(x) = x + 3y$ over the region bounded by $x + 3y = 0, x + 3y = 3, x - 3y = 0, x - 3y = 2$.

Solution: We use $s = x + 3y, t = x - 3y$. Then we have $x = \frac{s+t}{2}$ and $y = \frac{s-t}{6}$. Then the Jacobian is

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/6 & -1/6 \end{vmatrix} = -1/12 - 1/12 = -1/6.$$

Thus our integral is

$$\int_0^3 \int_0^2 s| - 1/6|\, dt\, ds = \int_0^3 \frac{s}{3}\, ds = \frac{s^2}{6}|_0^3 = \frac{3}{2}.$$

Problem 3 (S4)**.** Let R be the disk of radius 2 centered at the origin, with density $\rho(x, y) = x^2 + y^2 - 2x - 4y + 5$. What is the center of mass of R?

Solution: Because we're integrating over a disk, we want to use polar coordinates. We have $\rho(r, \theta) = r^2 - 2r\cos(\theta) - 4r\sin(\theta) + 5$, so the mass is

$$\begin{aligned}
m &= \int_0^2 \int_0^{2\pi} r^3 - 2r^2\cos(\theta) - 4r^2\sin(\theta) + 5r\, d\theta\, dr \\
&= \int_0^2 r^3\theta - 2r^2\sin(\theta) + 4r^2\cos(\theta) + 5r\theta\Big|_0^{2\pi} dr \\
&= \int_0^2 2\pi r^3 + 10\pi r\, dr = \frac{\pi}{2}r^4 + 5\pi r^2\Big|_0^2 = 8\pi + 20\pi = 28\pi.
\end{aligned}$$

Now we need to find the moments.

$$\begin{aligned}
M_x &= \int_0^2 \int_0^{2\pi} (r\sin(\theta))r^3 - 2r^2\cos(\theta) - 4r^2\sin(\theta) + 5r\, d\theta\, dr \\
&= \int_0^2 \int_0^{2\pi} r^4\sin(\theta) - 2r^3\sin(\theta)\cos(\theta) - 4r^3\sin^2(\theta) + 5r^2\sin(\theta)\, d\theta\, dr \\
&= \int_0^2 -r^4\cos(\theta) - r^3\sin^2(\theta) - r^3\left(2\theta - \sin(2\theta)\right) - 5r^2\cos(\theta)\Big|_0^{2\pi} dr \\
&= \int_0^2 -4\pi r^3\, dr = -\pi r^4\Big|_0^2 = -16\pi
\end{aligned}$$

and

$$\begin{aligned}
M_y &= \int_0^2 \int_0^{2\pi} (r\cos(\theta))r^3 - 2r^2\cos(\theta) - 4r^2\sin(\theta) + 5r\, d\theta\, dr \\
&= \int_0^2 \int_0^{2\pi} r^4\cos(\theta) - 2r^3\cos^2(\theta) - 4r^3\cos(\theta)\sin(\theta) + 5r^2\cos(\theta)\, d\theta\, dr \\
&= \int_0^2 r^4\sin(\theta) - 2r^3\left(\frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right) - 2r^3\sin^2(\theta) + 5r^2\sin(\theta)\Big|_0^{2\pi} dr \\
&= \int_0^2 -2\pi r^3\, dr = -\frac{1}{2}\pi r^4\Big|_0^2 = -8\pi.
\end{aligned}$$

Thus we compute

$$\begin{aligned}
\overline{x} &= \frac{M_y}{m} = \frac{-8\pi}{28\pi} = \frac{-2}{7} \\
\overline{y} &= \frac{M_x}{m} = \frac{-16\pi}{28\pi} = \frac{-4}{7}
\end{aligned}$$