5 Integration

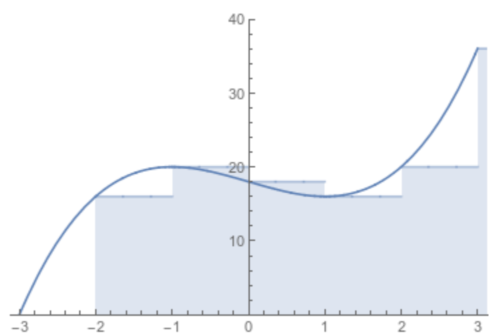
5.1 Riemann sums in multiple variables

Fundamentally, integrals are trying to add up all the value a function has in a given region. We do this by dividing the region up into a bucnh of subregions, estimating the total value in each subregion, and then adding these all back up.

In single-variable calculus we did this with a *Riemann Sum*. You might recall that we defined

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^n f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(a + i\frac{b-a}{n}\right) \cdot \frac{b-a}{n}.$$

The basic idea here is that we divide the interval $[a, b]$ up into n subintervals. Then we pick some point x_i^* in the subinterval to represent the "average" value in that interval, and estimate the total value to be $f(x_i^*)\Delta x$. We graphically represent this by drawing a rectangle for every subinterval with height $f(x_i^*)$, and adding up the areas of the rectangles.

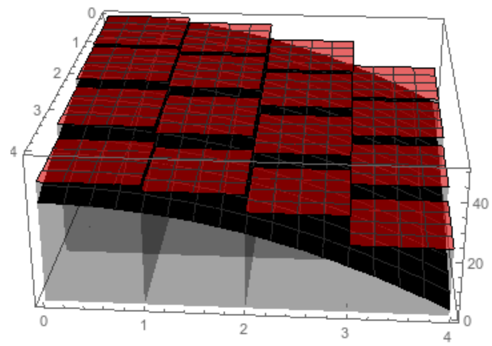


We'd like to do the same thing for a function of two or more variables. We'll stick with a two-variable function for now, and build the same picture. But since our function has two input variables, the geometry becomes three-dimensional. Rather than starting with an interval and dividing it into subintervals, we'll start with a rectangle and divide it into subrectangles.

Definition 5.1. Suppose $f(x, y)$ is continuous on a rectangle $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. Let (u_{ij}, v_{ij}) be any point in the ijth subrectangle. We define the *definite integral* of f over R to be

$$\int_R f\,dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(u_{ij}, v_{ij})\Delta x \Delta y.$$

If R is a non-rectangular region, we define $\int_R f\,dA$ similarly, except we ignore any subrectangle not contained in R. We can think of this as treating $f(u_{ij}, v_{ij}) = 0$ if (u_{ij}, v_{ij}) is not in R.



With the single-variable integral, we might worry that it matters *which* choice of value we take, but it turns out that that doesn't matter: in the limit they will converge to the same thing. The same is true in more variables.

Theorem 5.2. *If $f(x, y)$ is continuous and R is bounded, then $\int_R f dA$ converges, and the limit does not depend on the choices of (u_{ij}, v_{ij}).*

Sketch of Proof. If f is continuous on a closed and bounded region, then as Δx and Δy tend to zero, the difference between the maximum and minimum possible values of $f(x, y)$ within each rectangle tend to zero. Thus the largest possible sum and the smallest possible sum will converge to the same point; by the squeeze theorem, any intermediate sum will also converge. □

We can interpret this sum in a couple of different ways. One is volume. In the single-variable case, the integral estimates the (signed) area under the curve. In the multiple variable case, it estimates the volume under the surface given by the graph of the function.

Example 5.3. Suppose we want to estimate the area under the function $f(x, y) = 16 - 3x^2 - y^2$ on the rectangle with corners at $(0, 0)$ and $(2, 2)$. We can divide this up into four subrectangles, each of which is 1×1.

First let's get a definite overestimate, by always taking the highest point in each subrectangle. It's not too hard to see that for f, this will always be the point closest to the origin. So we have

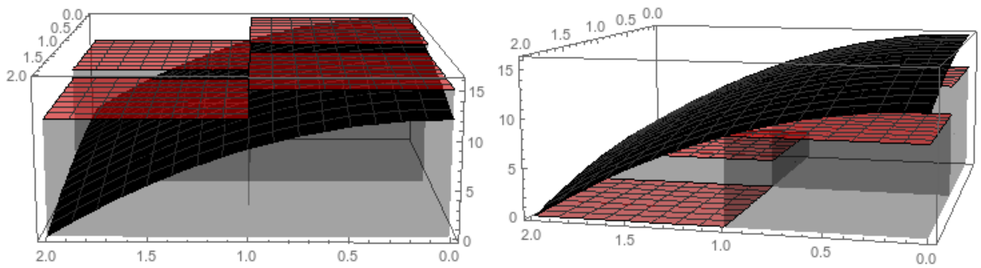
$$\int_R f da \approx f(0,0) \cdot 1 + f(1,0) \cdot 1 + f(0,1) \cdot 1 + f(1,1) \cdot 1 = 16 + 13 + 15 + 12 = 56.$$

We can also get an underestimate by taking the lowest value, which in this case will always be the upper-right point.

$$\int_R f da \approx f(1,1) \cdot 1 + f(2,1) \cdot 1 + f(1,2) \cdot 1 + f(2,2) \cdot 1 = 12 + 3 + 9 + 0 = 24.$$

So we can be pretty sure the volume is somewhere between 24 and 56. We would probably estimate something like $(24 + 56)/2 = 40$.

(If we compute the integral exactly, as we will learn in the next section, we will see that the integral is $\frac{128}{3} \approx 42.67$, so this estimate isn't too bad!)



"Volume under the surface" is a good way to interpret a 2-dimensional integral, but doesn't make much sense of a three-dimensional integral. (We can talk about the "hyper-volume" of the four-dimensional region, but that doesn't give much intuition since we can't really visualize hypervolumes).

Another way of understanding the integral is to think about averages. The integral $\int_R f dA$ is somehow computing the "total" value of f in the region. So we can also compute the *average* value of f in the region to be

$$\text{average} = \frac{1}{\text{Area}(R)} \int_R f dA.$$

This interpretation makes perfect sense in any number of variables we choose.

Definition 5.4. Suppose $f(x, y, z)$ is continuous on a region R, and let $(u_{ijk}, v_{ijk}, w_{ijk})$ be a point in the ijkth sub-prism. Then we define

$$\int_R f dV = \lim_{\Delta x, \Delta y, \Delta z \to 0} \sum_{i,j,k} f(u_{ijk}, v_{ijk}, w_{ijk}) \Delta x \Delta y \Delta z.$$

5.2 Iterated integrals

Computing multivariable integrals by writing out an expression for the Riemann sum and computing the limit is terrible. Fortunately we don't have to do that.

In single-variable calculus, we avoided doing the Riemann sum through the Fundamental Theorem of Calculus, which allowed us to evaluate an antiderivative on the endpoints of an interval, rather than summing the function on the whole interval. That is in fact possible to do here, but is somewhat complex, since the boundary of a two-dimensional region has infinitely many points. We'll return to this idea towards the end of the course. But for right now, we'll do something much simpler.

When we wrote down the definition of a two-variable Riemann sum, we just said to add up the values for all the subrectangles; we didn't say anything about what order to add them up in. And as long as the sum is finite, this can't possibly matter.

For infinite sums, the order you add things up in can matter (see e.g. the Riemann Series Theorem if you want to know more about this). But fortunately, it turns out that in this case it does not.

Theorem 5.5 (Fubini)**.** *Let $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\} = [a, b] \times [c, d]$, and let $f(x, y)$ be continuous on R. Then*

$$\int_R f dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

This means that rather than somehow doing the "whole" double integral, we can do two single-variable integrals in succession. And we already know how to do those!

Example 5.6. Let $R = \{(x, y) : 1 \leq x \leq 4, 0 \leq y \leq 3\}$ and let $f(x, y) = xy^2$. Then we can compute

$$\begin{aligned}\int_R f\, dA &= \int_1^4 \int_0^3 xy^2\, dy\, dx = \int_1^4 \left(xy^3/3|_0^3\right)\, dx \\ &= \int_1^4 9x\, dx = 9x^2/2|_1^4 = 72 - 9/2 = 135/2.\end{aligned}$$

Alternatively, we could compute:

$$\begin{aligned}\int_R f\, dA &= \int_0^3 \int_1^4 xy^2\, dx\, dy = \int_0^3 \left(x^2/2y^2|_1^4\right)\, dy \\ &= \int_0^3 8y^2 - y^2/2\, dy = \int_0^3 15y^2/2\, dy = 15y^3/6|_0^3 = 135/2.\end{aligned}$$

Notice we get the same answer with either order of integration.

Example 5.7 (Recitation)**.** Suppose we have a building with a corrugated sine-wave roof. It is 6 meters wide and 8 meters long. The corners are 2 and 3 meters high, and along the length the sine wave oscillates four times. What is the volume of the building?

The height is given by $f(x, y) = 2 + x/6 + \sin(\pi y)$. Then the volume is given by

$$\begin{aligned}
\int_0^6 \int_0^8 2 + x/6 + \sin(\pi y)\, dy\, dx &= \int_0^6 \left(2y + xy/6 - \cos(\pi y)/\pi\Big|_0^8\right)\, dx \\
&= \int_0^6 16 + 4x/3 - 1/\pi - (0 + 0 - 1/\pi)\, dx \\
&= \int_0^6 16 + 4x/3\, dx = 16x + 2x^2/3\Big|_0^6 = 96 + 24 = 120.
\end{aligned}$$

Integrals of three-variable functions work exactly the same way that integrals of two variables work. We just have three iterated integrals instead of two.

Example 5.8. Suppose we have a box that has a 3 inch square base, and is 4 inches tall, and has a density of $1 + xy + yz + xz^2$ ounces per cubic inch. What is the total mass?

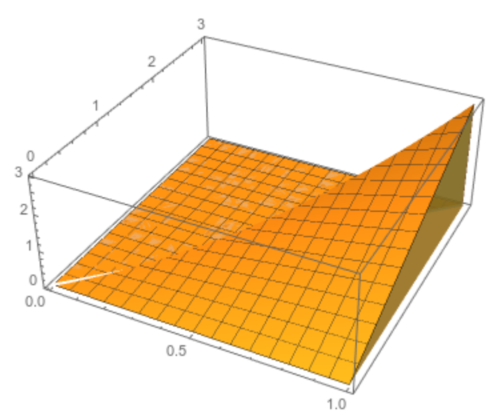
We want to compute the integral of $f(x, y, z) = 1 + xy + yz + xz^2$ over this rectangular box. So we compute

$$\begin{aligned}
M &= \int_0^3 \int_0^3 \int_0^4 1 + xy + yz + xz^2\, dz\, dy\, dx = \int_0^3 \int_0^3 z + xyz + yz^2/2 + xz^3/3\Big|_0^4\, dy\, dx \\
&= \int_0^3 \int_0^3 4 + 4xy + 8y + 64x/3\, dy\, dx = \int_0^3 4y + 2xy^2 + 4y^2 + 64xy/3\Big|_0^3\, dx \\
&= \int_0^3 12 + 18x + 36 + 64x\, dx = 48x + 41x^2\Big|_0^3 = 513.
\end{aligned}$$

Thus the box has a mass of 513 ounces.

We can also use iterated integrals to integrate over non-rectangular (or non-box) regions. In this case we'll let x (say) vary from its minimum possible value to its maximum possible value; but for each x, the possible y values will depend on the current x value.

Example 5.9. Integrate the function $f(x, y) = xy$ over the triangle with corners at $(0, 0)$, $(1, 0)$, and $(1, 3)$.



We have x varying from 0 to 1. The upper bound of the triangle is given by the line $y = 3x$, so the y bounds are from 0 to $3x$. Thus we have the double integral

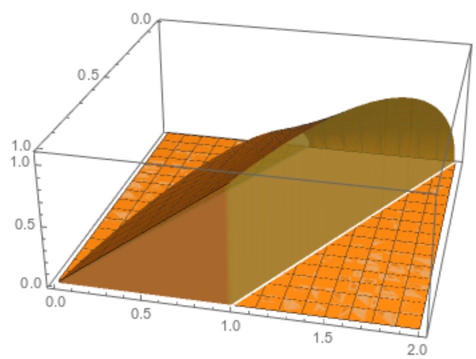
$$\int_0^1 \int_0^{3x} xy\,dy\,dx = \int_0^1 xy^2/2|_0^{3x}\,dx = \int_0^1 9x^3/2\,dx$$
$$= 9x^4/8|_0^1 = 9/8.$$

We could just as easily have done it the other way. y varies from 0 to 3, and x varies from $y/3$ to 1. So we have the double integral

$$\int_0^3 \int_{y/3}^1 xy\,dx\,dy = \int_0^3 x^2y/2|_{y/3}^1\,dy = \int_0^3 (y/2 - y^3/18)\,dy$$
$$= y^2/4 - y^4/72|_0^3 = 9/4 - 9/8 = 9/8.$$

Thus we get the same answer integrating either way.

Example 5.10. Let's integrate the function $f(x, y) = y\sqrt{x}$ over the parallelogram with corners at $(0, 1), (0, 2), (1, 0), (1, 1)$.



We see that x varies from 0 to 1, and y varies from $1 - x$ to $2 - x$. So we have

$$\int_0^1 \int_{1-x}^{2-x} y\sqrt{x}\,dy\,dx = \int_0^1 2y^2\sqrt{x}/2|_{1-x}^{2-x}\,dx = \int_0^1 (2-x)^2\sqrt{x}/2 - (1-x)^2\sqrt{x}/2\,dx$$
$$= \int_0^1 (4 - 4x + x^2 - 1 + 2x - x^2)\sqrt{x}/2\,dx = \int_0^1 3/2\sqrt{x} - x^{3/2}\,dx$$
$$= x^{3/2} - 2/5x^{5/2}|_0^1 = 3/5.$$

Could we integrate the other way? Sure. But it's actually a big pain, since writing x as a function of y would have to go piecewise: we'd get something like

$$x = \begin{cases} 1 - y \leq x \leq 1 & y \leq 1 \\ 0 \leq x \leq 2 - y & 1 \leq y \leq 2 \end{cases}$$

So we'd have to set up and evaluate two separate integrals here, and get something like

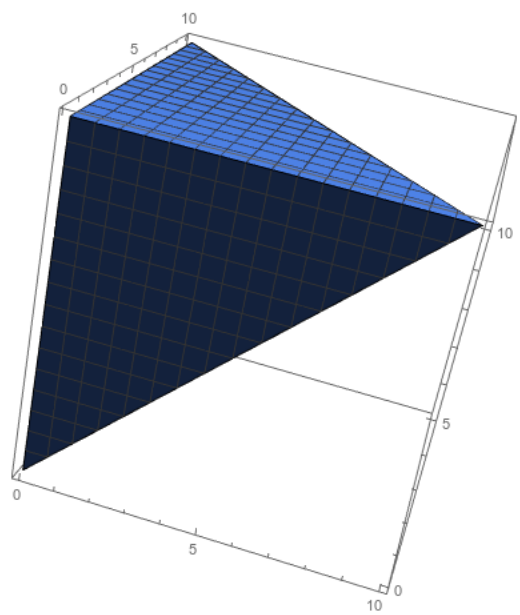
$$\int_0^1 \int_{1-y}^1 y\sqrt{x}\,dx\,dy + \int_1^2 \int_0^{2-y} y\sqrt{x}\,dy\,dx.$$

Integrating by y and then x is very much the correct choice here.

Remark 5.11. Whenever setting up an iterated integral, remember that the final answer should be a number. Therefore the bounds of the outer integral should always be constants. The bounds on the inner integrals can depend on variables from integrals to the outside, but not on variables from integrals to the inside.

At each step, you should have one fewer variable to worry about (although possibly a more complex algebraic expression).

Example 5.12. Find the volume of the region bounded by $z = x+y$, $z = 10$, and the planes $x = 0, y = 0$.



We can set this up as a two-variable integral or as a three-variable integral. As a two-variable integral we'd need the region in the plane and the height. The solid exists over a region bounded by $0 \leq x \leq 10$ and $0 \leq y \leq 10-x$. Then the height is given by the difference between $z = 10$ and $z = x + y$, so we have $f(x, y) = 10 - x - y$. Then we get the integral

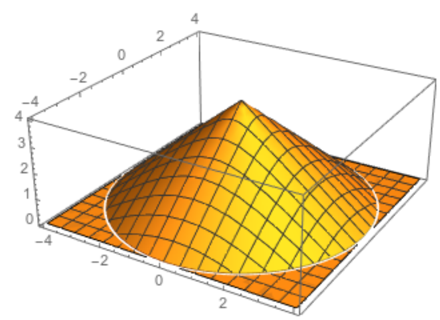
$$\begin{aligned}\int_0^{10} \int_0^{10-x} 10 - x - y\, dy\, dx &= \int_0^{10} 10y - xy - y^2/2|_0^{10-x}\, x \\ &= \int_0^{10} 100 - 10x - 10x + x^2 - (100 - 20x + x^2)/2\, dx \\ &= \int_0^{10} 50 - 10x + x^2/2\, dx = 50x - 5x^2 + x^3/6|_0^{10} \\ &= 500 - 500 + 500/6 = 500/6.\end{aligned}$$

But it's actually a bit more natural to express this as a triple integral. The volume of a region is just the integral of the function 1 over that region. So we can write

$$\begin{aligned}V &= \int_0^{10} \int_0^{10-x} \int_{x+y}^{10} dz\, dy\, dx \\ &= \int_0^{10} \int_0^{10-x} z|_{x+y}^{10}\, dy\, dz \\ &= \int_0^{10} \int_0^{10-x} 10 - x - y\, dy\, dx.\end{aligned}$$

This of course gets us the same answer as before, but is often a bit easier to think about.

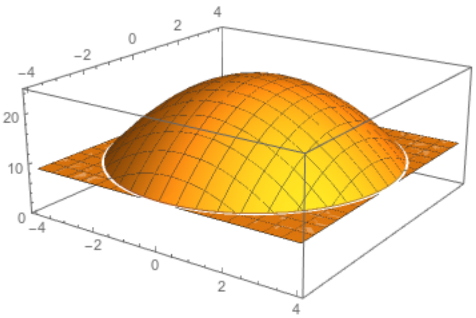
Example 5.13. Set up an integral to find the mass of a solid cone bounded by the xy plane and the cone $z = 4 - \sqrt{x^2 + y^2}$, if the density is given by $\delta(x, y, z) = xz$.



We have the iterated integral

$$\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_0^{4-\sqrt{x^2+y^2}} xz\, dz\, dy\, dx.$$

Example 5.14 (Recitation)**.** Set up an integral to find the volume of the solid below the graph of $f(x, y) = 25 - x^2 - y^2$ and above the plane $z = 9$.



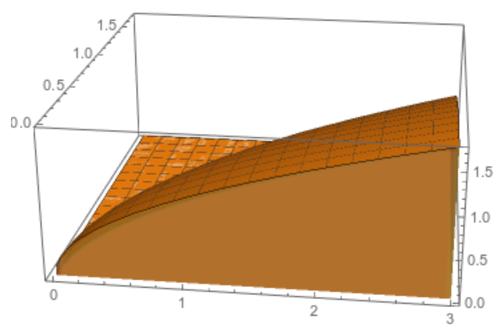
The two surfaces intersect where $x^2 + y^2 = 16$. We can either write the double integral

$$\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 16 - x^2 - y^2 \, dy \, dx$$

or we can write the triple integral

$$\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{9}^{25-x^2-y^2} \, dz \, dy \, dx$$

Example 5.15. Set up an integral to find the volume of the region in the first octant bounded by the coordinate planes, the plane $z = 3$, and the surface $z = x^2 + y^2$.



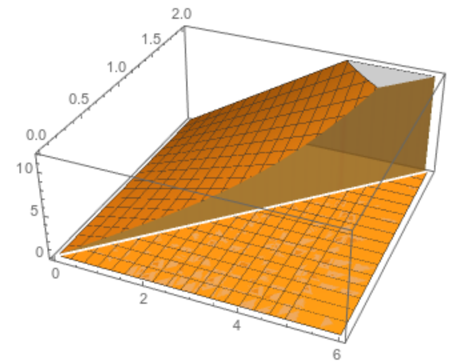
We can see we have z varying from 0 to 3. For each z, we have x varying from 0 to $\sqrt{z}$, and then y varying from 0 to $\sqrt{z - x^2}$. So we get the integral

$$\int_0^3 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} 1 \, dy \, dx \, dz.$$

In most of these cases we have a few different options for how to set up the integral. So far these choices haven't mattered that much, but sometimes they matter a great deal.

Example 5.16.

$$\int_0^6 \int_{x/3}^{2} x\sqrt{y^3 + 1} \, dy \, dx.$$

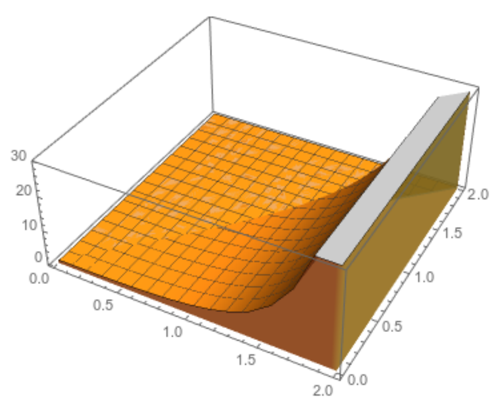


The integral with respect to y is a huge pain, so we don't do it. We sketch the region: x goes from 0 to 6, and y goes from $x/3$ to 2. We can turn this around to say: y goes from 0 to 2, and x goes from 0 to $3y$. So we get

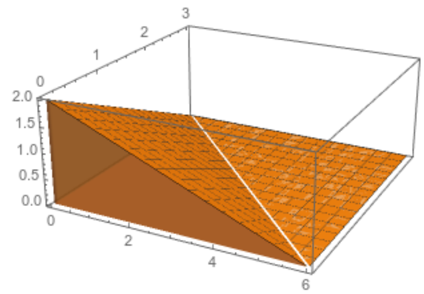
$$\begin{aligned}\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\, dx\, dy &= \int_0^2 \left(x^2/2\sqrt{y^3+1}\Big|_0^{3y} \right) dy \\ &= \int_0^2 \left(9y^2/2\sqrt{y^3+1} \right) dy \\ &= (y^3+1)^{3/2}\big|_0^2 = 27 - 1 = 26.\end{aligned}$$

Example 5.17 (Recitation)**.**

$$\begin{aligned}\int_0^2 \int_y^2 e^{x^2}\, dx\, dy &= \int_0^2 \int_0^x e^{x^2}\, dy\, dx \\ &= \int_0^2 xe^{x^2}\, dx = e^{x^2}/2\big|_0^2 = e^4/2 - 1/2.\end{aligned}$$



Example 5.18 (Bonus)**.** Find the mass of the solid bounded by the xy plane, the yz plane, the xz plane, and the plane $x + 3y + 2z = 6$, if the density is given by $\delta(x, y, z) = x + z$.

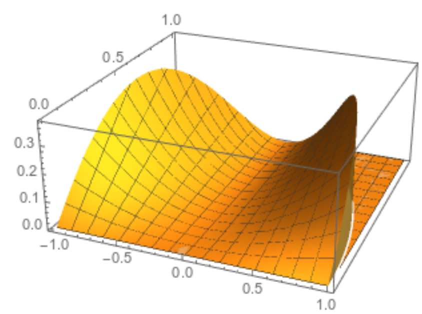


We see that x varies from 0 to 6, and then z varies from 0 to $(6-x)/2$, and then y varies from 0 to $(6-x-2z)/3$. So we get

$$M = \int_0^6 \int_0^{3-x/2} \int_0^{2-x/3-2z/3} x + z \, dy \, dz \, dx = 27/2.$$

And sometimes, no matter what you do, the integral will be gross.

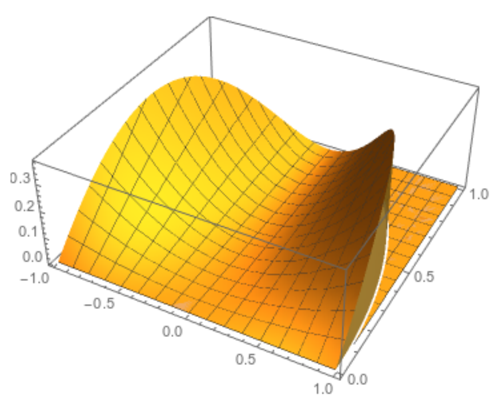
Example 5.19. Integrate $f(x,y) = x^2 y$ over the upper half of the unit circle.



We have that $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$. So we get

$$\begin{aligned}
\int_{-1}^1 \int_0^{\sqrt{1-x^2}} x^2 y \, dy \, dx &= \int_{-1}^1 x^2 y^2/2|_0^{\sqrt{1-x^2}} \, dx \\
&= \int_{-1}^1 x^2(1-x^2)/2 \, dx = \int_{-1}^1 x^2/2 - x^4/2 \, dx \\
&= x^3/6 - x^5/10|_{-1}^1 = 1/6 - 1/10 + 1/6 - 1/10 = 2/15.
\end{aligned}$$

Example 5.20. Integrate $f(x,y) = x^2 y^2$ over the upper half of the unit circle.



We have that $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$. So we get

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx = \int_{-1}^{1} x^2y^3/3|_0^{\sqrt{1-x^2}}\,dx$$
$$= \int_{-1}^{1} x^2(1-x^2)^{3/2}/3\,dx$$

and this has suddenly become a huge mess—much worse than the previous problem. We can use trigonometric substitution plus some grindy arguments to find that this is equal to

$$\frac{1}{144}\left(x\sqrt{1-x^2}(-8x^4+14x^2-3)+3\arcsin(x)\right)\Big|_{-1}^{1} = \frac{\pi}{48},$$

but ultimately there's nothing we can do to this integral that will make it nice.

The fundamental problem in this last exampleis that since we're integrating over a circle, we have these $\sqrt{1-x^2}$ terms that we just can't get rid of.

Unless we develop a completely different approach to setting up integrals, that somehow is more compatible with circles.

5.3 Integrals in Polar Coordinates

Describing circles in Cartesian coordinates is fundamentally a bit awkward. It's much easier to describe a circle or circle-like region in terms of polar coordinates.

Definition 5.21. The *polar coordinates* of a point $P \in \mathbb{R}^2$ are a pair of numbers (r,θ), where r is the distance between P and the origin O, and θ is the angle between the vector $\vec{i}$ and the vector $\overrightarrow{OP}$.

We always choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

Proposition 5.22. *Suppose (x,y) are the cartesian coordinates of a point P, and $(r\, theta)$ are the polar coordinates. Then:*

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2+y^2}$
- $\theta = \pm\arctan y/x.$

Example 5.23. The polar equation for a circle of radius c is $r = c$. The closed disk of radius c is given by the set $\{(r, c) : 0 \leq r \leq c, 0 \leq \theta < 2\pi\}$. The Cartesian coordinates are $\{(x, y) : x^2 + y^2 \leq c^2\}$.

The wedge of the closed unit disk in the first (upper-right) quadrant is $\{(r, \theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\}$. The Cartesian coordinates are $\{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1.\}$

The set $\{(r, \theta) : 1 \leq r \leq 2, \pi \leq \theta \leq 3\pi/2\}$ is a wedge of an annulus with inner radius 1 and outer radius 2, in the third (lower-left) quadrant. The Cartesian coordinates here are $\{(x, y) : x \leq 0, y \leq 0, 1 \leq x^2 + y^2 \leq 4\}$.

The polar equation for the line $y = 2x$ is $r \sin\theta = 2r \cos\theta$, which reduces to $\sin\theta = 2\cos\theta$.

Notice that all the circle equations become much simpler than their cartesian equivalents, but the line (and anything else rigid and rectangular) becomes much more complex.

We want to exploit this complexity reduction to make integrals of functions over circular regions easier. When we integrated over a rectangular region, we did this by dividing the region into rectangles. Using polar coordinates to integrate over a circular or wedge-like region, we'll divide the region into *subwedges.*

What is the area of a wedge? Each wedge is *roughly* a rectangle. (This is very rough, but in the limit it all washes out). The thickness of the rectangle is the change in the radius, so we call that dr. The width of the rectangle is *proportional* to the change in angle, but not equal to it: by definition, an arc of θ radians has a length of θr. Thus the width of our rectangle is $r\,d\theta$, the change in the angle times the actual radius.

This if we want to integrate a function in polar coordinates, we use the formula

$$I = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta.$$

Note the extra r in the formula! This is very important, and converts a number of integrals from "obnoxious" to "easy".

Example 5.24. Let's integrate $f(x, y) = x^2 y$ over the upper half of the unit circle.

We see that this is a region given by $0 \leq r \leq 1$ and $0 \leq \theta \leq \pi$. So we compute

$$\begin{aligned} I &= \int_0^1 \int_0^\pi r^2 \cos^2\theta r \sin\theta r\, d\theta\, dr \\ &= \int_0^1 \int_0^\pi r^4 \cos^2\theta \sin\theta\, d\theta\, dr \\ &= \int_0^1 r^4 \frac{-1}{3} \cos^3\theta|_0^\pi\, dr = \int_0^1 r^4 \frac{2}{3}\, dr \\ &= \frac{2}{15} r^5|_0^1 = \frac{2}{15}. \end{aligned}$$

Example 5.25. What about $f(x, y) = x^2y^2$ over that same region? We have

$$
\begin{aligned}
I &= \int_0^1 \int_0^\pi r^2 \cos^2\theta r^2 \sin^2\theta r \, d\theta \, dr \\
&= \int_0^1 \int_0^\pi r^5 \cos^2\theta \sin^2\theta \, d\theta \, dr \\
&= \int_0^1 r^5 \left(\frac{\theta}{8} - \frac{1}{32}\sin(4\theta) \right)\Big|_0^\pi \, dr \\
&= \int_0^1 \frac{\pi r^5}{8} \, dr = \frac{\pi r^6}{48}\Big|_0^1 = \frac{\pi}{48}.
\end{aligned}
$$

And while I didn't actually show the work to do that first antiderivative, it's a standard calc 2 trick—unlike the non-polar version, which is basically undoable.

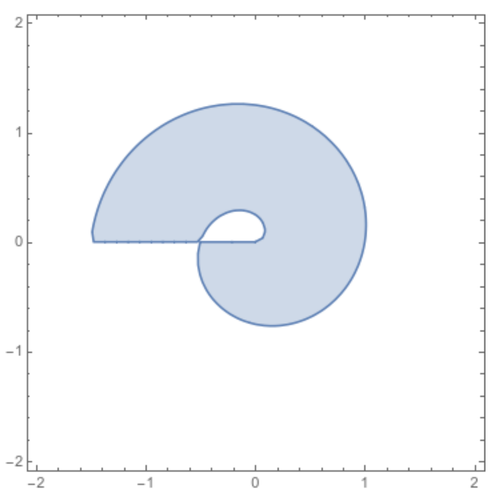
Some functions also become much easier to integrate in polar coordinates.

Example 5.26. Integrate the function $f(x, y) = (x^2 + y^2)^{-1/2}$ over the annulus with inner radius 1 and outer radius 2.

We have bounds $1 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. More importantly, we see that $f(x, y) = (x^2 + y^2)^{-1/2} = (r^2)^{-1/2} = \frac{1}{r}$. Thus we have

$$
\begin{aligned}
I &= \int_0^{2\pi} \int_1^2 \frac{1}{r} \cdot r \, dr \, d\theta = \int_0^{2\pi} r|_1^2 \, d\theta \\
&= \int_0^{2\pi} 1 \, d\theta = 2\pi.
\end{aligned}
$$

Example 5.27. Let's find the area of the spiral that has thickness 1, and has inner radius going from 0 to 1 over one complete rotation.



$$\begin{aligned}\int_0^{2\pi}\int_{\theta/(2\pi)}^{1+\theta/(2\pi)} r\,dr\,d\theta &= \int_0^{2\pi} \left.\frac{r^2}{2}\right|_{\theta/(2\pi)}^{1+\theta/(2\pi)} d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (1+\theta/(2\pi))^2 - (\theta/(2\pi))^2\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 1+\theta/\pi+\theta^2/(4\pi^2) - \theta^2/(4\pi^2)\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 1+\theta/\pi\,d\theta = \frac{1}{2}\left(\theta+\theta^2/(2\pi)|_0^{2\pi}\right) \\ &= \frac{1}{2}\left(2\pi+(2\pi)^2/(2\pi)\right) = 2\pi.\end{aligned}$$

5.4 Cylindrical and Spherical Coordinates

We can extend this idea to three dimensions. There are two different ways to do this, which are suited to different types of regions.

Definition 5.28. The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ are a triple of numbers (r, θ, z), where r is the distance between the origin O and the projection of P into the xy plane; and θ is the angle between the vector $\vec{i}$ and the projection of $\overrightarrow{OP}$ into the xy plane; and z is the height.

We always choose these numbers so that r is positive, and $\theta \in [0, 2\pi)$.

Proposition 5.29. *Suppose* (x, y, z) *are the cartesian coordinates of a point* P*, and* $(r\,theta, h)$ *are the polar coordinates. Then:*

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r = \sqrt{x^2+y^2}$
- $\theta = \pm\arctan y/x$
- $z = h.$

We can work out the integral formula here, just like we did for polar integrals. We divide our region into three-dimensional wedges—imagine a wedge of cheese. Each wedge is *roughly* a rectangular prism, as in polar integrals. The area of the base of the wedge is still $r\,dr\,d\theta$, and the height is dz, so when we do our integrals in cylindrical coordinates, we integrate $f(r, \theta, z)r\,dr\,d\theta\,dz$.

Example 5.30. Integrate xz over wedge cut from cylinder 4 cm high and 6 cm in radius, angle $\pi/6$ above x axis.

$$\int_0^4 \int_0^6 \int_0^{\pi/6} r\cos\theta zr\,d\theta\,dr\,dz = 288$$

Example 5.31. Integrate the function xyz over the cone bounded by $0 \leq z \leq 4$ and $x^2 + y^2 = z^2$ and the plane $z = 0$.

$$\int_0^4 \int_0^z \int_0^{2\pi} r^3 \cos\theta \sin\theta z\,d\theta\,dr\,dz = 0$$

Example 5.32. Set up an integral in cylindrical coordinates to find the volume inside the unit sphere.

$$\int_0^{2\pi} \int_{-1}^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} dz\,dr\,d\theta.$$

As we can see, cylindrical coordinates are still pretty unsuited to describing actual spheres. For those, we want to use a different coordinate system entirely.

Definition 5.33. The *spherical coordinates* of a point $P \in \mathbb{R}^3$ are a triple of numbers (ρ, θ, ϕ), where ρ is the distance between the origin O and the point P; θ is the angle between the vector $\vec{i}$ and the projection of $\overrightarrow{OP}$ into the xy plane; and ϕ is the angle between the vector $\overrightarrow{OP}$ and the vector $\vec{k}$.

We always choose these numbers so that ρ is positive, $\theta \in [0, 2\pi)$, and $\phi \in [0, \pi]$.

Proposition 5.34. *Suppose (x, y, z) are the cartesian coordinates of a point P, and $(r\, theta, h)$ are the polar coordinates. Then:*

- $x = \rho\sin(\phi)\cos(\theta)$
- $y = \rho\sin(\phi)\sin(\theta)$
- $z = \rho\cos(\phi)$
- $\rho^2 = x^2 + y^2 + z^2$.

Next we need the integral formula. We again divide our region into wedges, but these are wedges of a spherical shell, rather than the blocks-of-cheese that feature in cylindrical coordinates.

Again the thickness is just $d\rho$. We need to compute the area of the inner square of the wedge. We see that the "height" is determined by the length of the ϕ arc, and thus is $\rho\, d\phi$.

The "width" is given by the length of the θ arc. In cylindrical coordinates this was given by $r\, d\theta$, but we'll have something smaller in spherical coordinates: as you move away from the $z = 0$ plane (which is also the $\phi = \pi/2$ plane!) the radius of the circle given by intersecting the plane $z = z_0$ with the sphere $\rho = \rho_0$ will decrease, proportionately to $\sin \phi$. Thus the height of the wedge is $\rho \sin \phi\, d\theta$, and our integral is

$$\int_{\rho_1}^{\rho_2} \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} f\left(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi)\right) \rho^2 \sin \phi \, d\phi \, d\theta \, d\rho.$$

Example 5.35. Let's find the volume of the unit sphere. We have

$$\begin{aligned}
\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho &= \int_0^1 \int_0^{2\pi} -\rho^2 \cos(\phi)|_0^{\pi}\, d\theta\, d\rho \\
&= \int_0^1 \int_0^{2\pi} 2\rho^2\, d\theta\, d\rho \\
&= \int_0^1 4\pi\rho^2\, d\rho \\
&= \frac{4}{3}\pi\rho^3|_0^1 = 4\pi/3.
\end{aligned}$$

Example 5.36. Find the mass of a sphere with radius 3 and density equal to $\rho \cos^2 \theta$.

$$\begin{aligned}
\int_0^3 \int_0^{2\pi} \int_0^{\pi} \rho^3 \cos^2 \theta \sin \phi\, d\phi\, d\theta\, d\rho &= \int_0^3 \int_0^{2\pi} \rho^3 \cos^2 \theta (-\cos \phi)|0^{\pi}\, d\theta\, d\rho \\
&= \int_0^3 \int_0^{2\pi} 2\rho^3 \cos^2 \theta\, d\theta\, d\rho \\
&= \int_0^3 \rho^3 \left(\theta + \frac{\sin(2\theta)}{2} \right)\Big|_0^{2\pi} d\rho \\
&= \int_0^3 2\pi\rho^3\, d\rho = \pi\rho^4/2|_0^3 = \frac{81\pi}{2}.
\end{aligned}$$

5.5 Change of Coordinates in Integrals

In the last couple sections we looked at new coordinate systems: in section 5.3 we looked at a new coordinate system on $\mathbb{R}^2$, and in section 5.4 we looked at two different new coordinate systems in $\mathbb{R}^3$.

But there's nothing *really* special about these coordinate systems, except that circles, cylinders, and spheres come up frequently and are extremely annoying in Cartesian coordinates. But we can come up with plenty of other coordinate systems.

Example 5.37. Here are some coordinate systems we can put on $\mathbb{R}^2$:

$$(x, y) = T(s, t) = (s, t)$$
$$(x, y) = T(s, t) = (s\cos(t), s\sin(t))$$
$$(x, y) = T(s, t) = (t, s)$$
$$(x, y) = T(s, t) = (3s, s - t).$$

The first coordinate system is regular Cartesian coordinates, and the second is just polar coordinates. But the other two are new coordinate systems.

We can think about these functions as coordinate systems, or as transformations of $\mathbb{R}^2$. It's often helpful to think about what they do to a small rectangle. The transformation $T(s, t) = (t, s)$ will flip things across the line $y = x$. The second transformation will distort it into a parallelogram.

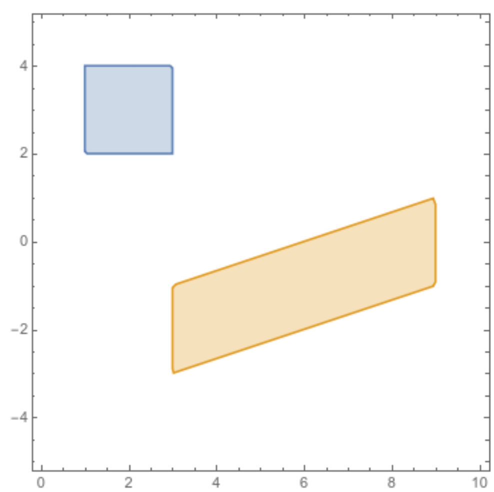


Figure 5.1: The transformation $T(s, t) = (3s, s - t)$ will convert a rectangle into a parallelogram

Remark 5.38. This is the same idea as "change of basis" in a linear algebra context.

In general, we can use customized coordinate system to make double integrals easier. We've actually seen this principle already in single-variable calculus: if we have a change of variables $x = g(u)$, then we can compute

$$\int_{g(a)}^{g(b)} f(u)\,du = \int_a^b f(g(x)) \cdot g'(x)\,dx.$$

We want to get a similar formula in 2 (or more) variables.

If we have a transformation $(x, y) = T(s, t) = (x(s, t), y(s, t))$, this means we'll divide our region up into rectangles in (s, t) coordinates, and sum up the values in each rectangle. To make this useful, we need to figure out how area in (s, t)-coordinates relates to area in (x, y) coordinates.

Suppose we have a a small rectangle in (s, t) coordinates, with corners at

$$(s, t), (s + \Delta s, t), (s, t + \Delta t), (s + \Delta s, t + \Delta t).$$

Then its image under the transformation T is going to be *some* four-sided shape with curved sides, with corners at the points

$$(x(s, t), y(s, t)), (x(s+\Delta s, t), y(s+\Delta s, t)), (x(s, t+\Delta t), y(s, t+\Delta t)), (x(s+\Delta s, t+\Delta t), y(s+\Delta s, t+\Delta t)).$$

When $\Delta s, \Delta t$ are small, we can treat this as a parallelogram. So we just need to find the area of a parallelogram.

You might recall from section 1.4 proposition 1.40 that the area of a parallelogram with sides given by the vectors $\vec{u}$ and $\vec{v}$ is $\|\vec{u} \times \vec{v}\|$. So we need to figure out the vectors for the sides of this parallelogram.

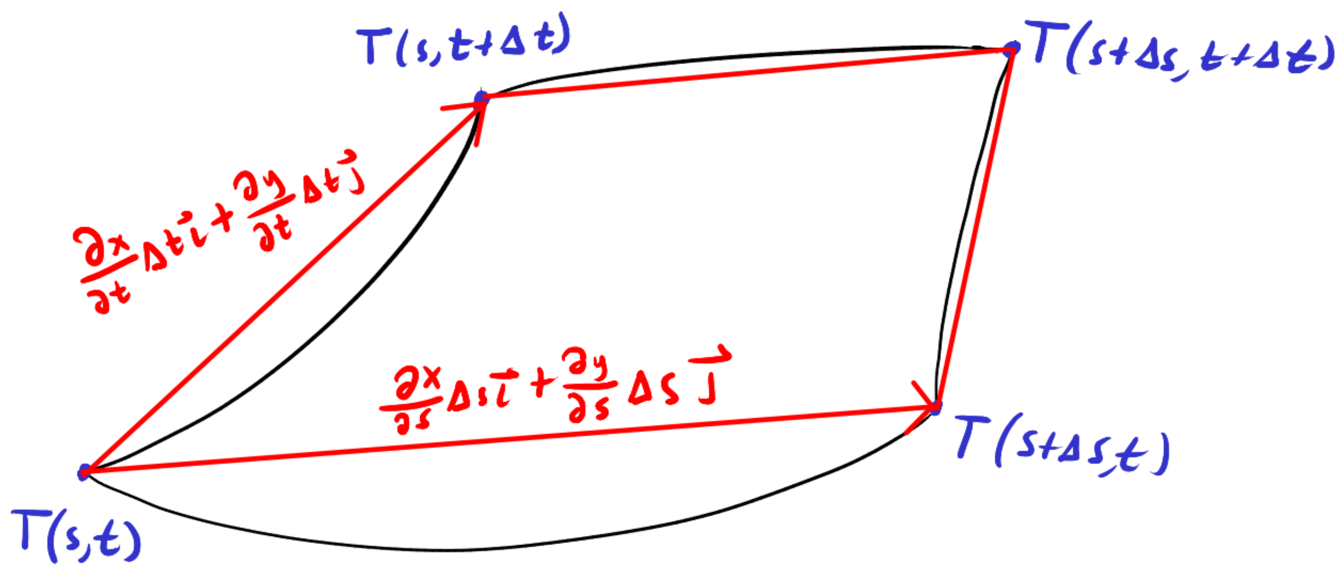


Figure 5.2: The image of a small rectangle under the transformation T

But since these vectors are just $\frac{\Delta x}{\Delta s}\vec{i} + \frac{\Delta y}{\Delta s}\vec{j}$ and $\frac{\Delta x}{\Delta t}\vec{i} + \frac{\Delta y}{\Delta t}\vec{j}$, we can approximate these with directional derivatives. In particular, when the sides are small, they are approximately given by $\frac{\partial x}{\partial s}\Delta s\vec{i} + \frac{\partial y}{\partial s}\Delta s\vec{j}$ and $\frac{\partial x}{\partial t}\Delta t\vec{i} + \frac{\partial y}{\partial t}\Delta t\vec{j}$. Thus the area of the parallelogram is approximately

$$\begin{aligned}\left|\left(\frac{\partial x}{\partial s}\Delta s\vec{i} + \frac{\partial y}{\partial s}\vec{j}\right) \times \left(\frac{\partial x}{\partial t}\Delta t\vec{i} + \frac{\partial y}{\partial t}\Delta t\vec{j}\right)\right| &= \left|\frac{\partial x}{\partial s}\Delta s\frac{\partial y}{\partial t}\Delta t - \frac{\partial x}{\partial t}\Delta t\frac{\partial y}{\partial s}\Delta s\right| \\ &= \left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial d} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right|\Delta s\Delta t.\end{aligned}$$

Definition 5.39. The *Jacobian* of a function is the determinant of the matrix of partial derivatives. Thus the Jacobian of $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(s,t) = \Big(x(s,t), y(s,t)\Big)$, is

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t}\end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}.$$

Thus in this notation, the area of the parallelogram we're studying is approximately $\left|\frac{\partial(x,y)}{\partial(s,t)}\right|\Delta s\Delta t$.

So now let's return to thinking about our integral. If we want to compute an integral, we have the following computation:

$$\begin{aligned}\int_{R_{x,y}} f(x,y)\,dA &= \lim \sum f(u^*_{ij}, v^*_{ij})\left|\frac{\partial(x,y)}{\partial(s,t)}\right|\Delta s\Delta t \\ &= \lim \sum f(x(s^*_{ij}, t^*_{ij}), y(s^*_{ij}, t^*_{ij}))\left|\frac{\partial(x,y)}{\partial(s,t)}\right|\Delta s\Delta t \\ &= \int_{R_{s,t}} f(x(s,t), y(s,t))\left|\frac{\partial(x,y)}{\partial(s,t)}\right|\,ds\,dt.\end{aligned}$$

Thus in summary, we can compute integrals in a new coordinate system by doing the following:

1. Substitue $x(s,t)$ and $y(s,t)$ for x and y in the inside of the integral.
2. Change the region/bounds to be described in terms of s and t.
3. Make the substitution $dx\,dy = \left|\frac{\partial(x,y)}{\partial(s,t)}\right|\,ds\,dt$.

Remark 5.40. This is in fact a generalization of u-substitution in single-variable calculus. Our Jacobian term $\frac{\partial(x,y)}{\partial(s,t)}$ is playing the role of $g'(x)$, and generally $g'(x)\,dx = du$ is the same idea as $\frac{\partial(x,y)}{\partial(s,t)}\,ds\,dt = dx\,dy$.

And we also have a change of bounds in both setups. In single-variable u-substitution we had $x = g(u)$ and thus changed our bounds to $g(a)$ and $g(b)$; in multivariable settings we have to do something more complicated, but it's the same basic idea.

Example 5.41. We can recover polar coordinates from this setup. Polar coordinates give the parametrization $x = r\cos\theta, y = r\sin\theta$. Then we compute

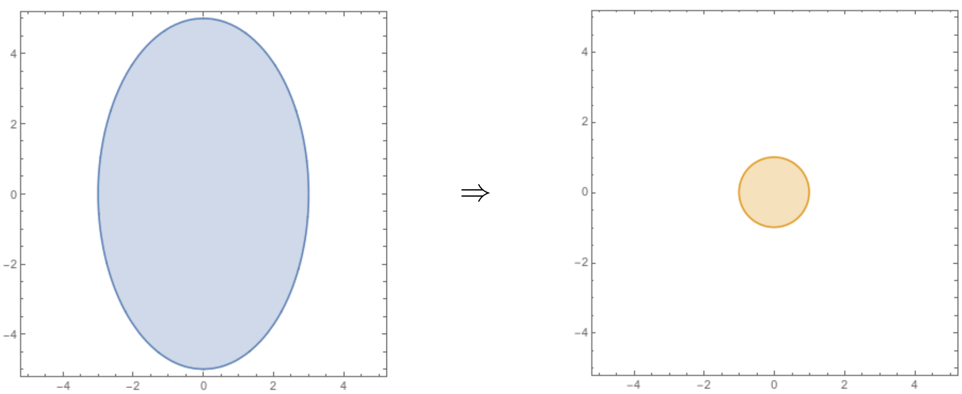
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

This gives us back exactly the conversion factor that we got before.

Example 5.42. Let's find the area of an ellipse given by the equation $x^2/a^2 + y^2/b^2 = 1$.

We could just do this in polar coordinates, but it would be a little messy, since an ellipse isn't actually a circle. So we can choose coordinates that turn it into a circle: if we take $x = as, y = bt$, then the equation becomes $s^2 + t^2 = 1$, so the region is the unit circle in the st plane. We calculate the Jacobian is $\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$. Thus the area of the ellipse is

$$\int_R 1\,dx\,dy = \int_T 1ab\,ds\,dt = ab\int_T 1\,ds\,dt = ab\pi.$$



We avoid actually computing an integral here, since we already know the area of the unit circle.

Example 5.43. Evaluate $\int_R x+y\,dA$ where R is the region with vertices $(0,0), (5,0), (5/2,5/2)$, and $(5/2,-5/2)$.

We can calculate that the equations for the boundary lines are $y = x, y = -x, y = x-5$, and $y = 5-x$.

There are two basic ways we could approach this. One is to set up a pair of double integrals with x, y coordinates: we get

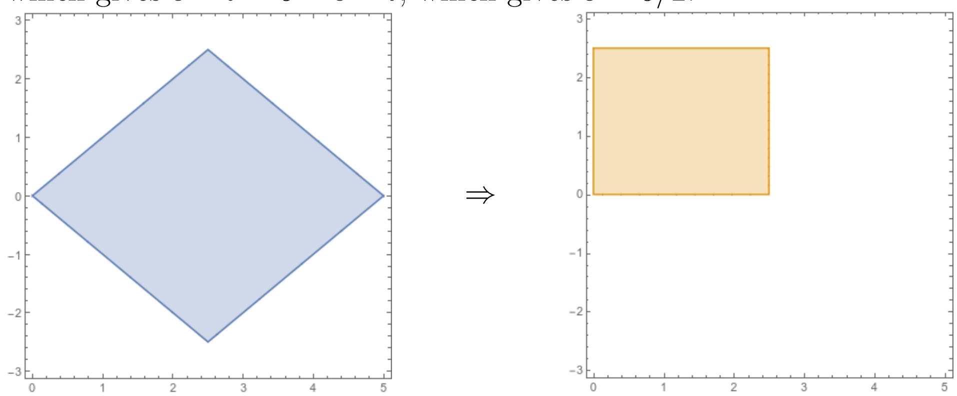
$$I = \int_0^{5/2}\int_{-x}^{x} x+y\,dy\,dx + \int_{5/2}^{5}\int_{x-5}^{5-x} x+y\,dy\,dx.$$

But this is long and annoying.

The other thing we can do is use a change of coordinates to convert this into a reasonable rectangle. We see that the region isn't particularly aligned in the directions of $\vec{i}$ and $\vec{j}$, but rather in the directions $\vec{i}+\vec{j}$ and $\vec{i}-\vec{j}$. So we might try a parametrization $x = s+t$ and $y = s-t$.

To find our new bounds we plug this into our boundary equations. For $y = x$ we get $s - t = s + t$, which gives us $t = 0$. For $y = -x$ we get $s - t = -s - t$, which gives us $s = 0$.

Similarly, for $y = x - 5$ we get $s - t = s + t - 5$. Solving gives $t = 5/2$. Finally, we have $y = 5 - x$, which gives $s - t = 5 - s - t$, which gives $s = 5/2$.



Thus, rather than having a complicated integral setup, we just get bounds $0 \leq s \leq 5/2, 0 \leq t \leq 5/2$. Our integrand is $x + y = s + t + s - t = 2s$. And our Jacobian is

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = |-1-1| = 2.$$

Thus we have the integral

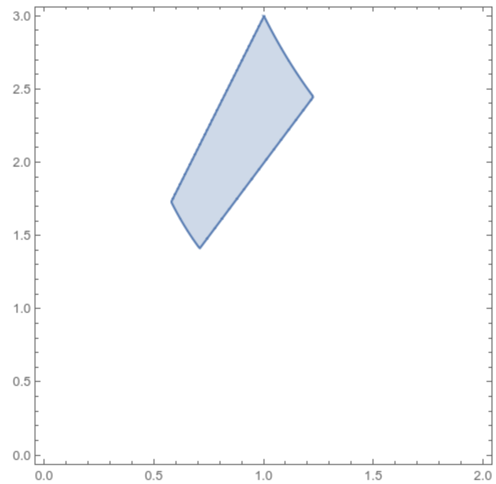
$$\begin{aligned} I &= \int_0^{5/2}\int_0^{5/2} 2s \cdot 2\,dt\,ds \\ &= \int_0^{5/2} 4st|_0^{5/2}\,ds = \int_0^{5/2} 10s\,ds \\ &= 5s^2|_0^{5/2} = \frac{125}{4}. \end{aligned}$$

Example 5.44. Let's find the area of the region bounded by $xy = 1, xy = 3, y = 2x, y = 3x$.

We could try to set up this integral in cartesian coordinates, but it sounds extremely unpleasant. But we can transform this into a rectangle by picking better coordinates. I'll take $s = xy$, so that s will go from 1 to 3. Then I'll take $t = y/x$, so that t will go from 2 to 3. And I can write $x = \sqrt{s/t}$ and $y = \sqrt{st}$.

To compute the Jacobian, we have

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{1}{2\sqrt{st}} & \frac{-\sqrt{s}}{2\sqrt{t^3}} \\ \frac{\sqrt{t}}{2\sqrt{s}} & \frac{\sqrt{s}}{2\sqrt{t}} \end{vmatrix} = \frac{1}{4t} - \frac{-1}{4t} = \frac{1}{2t}.$$



So we have

$$\begin{aligned}\int_R 1\,dy\,dx &= \int_1^3\int_2^4 1\cdot\left|\frac{1}{2t}\right|\,dt\,ds\\ &= \int_1^3\int_2^4 \frac{1}{2t}\,dt\,ds\\ &= \int_1^3 \frac{1}{2}\ln|t|\Big|_2^4\,ds = \int_1^3 \frac{1}{2}\left(\ln(4)-\ln(2)\right)\,ds\\ &= \ln(4)-\ln(2) = \ln(2).\end{aligned}$$

So the region has area $\ln(2)$.

Example 5.45. We can also generalize this to three variables. Spherical coordinates are given by the transformation $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$. Then we compute the Jacobian is

$$\begin{aligned}\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix} &= \begin{vmatrix}\sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta\\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta\\ \cos\phi & 0 & -\rho\sin\phi\end{vmatrix}\\ &= |-\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin^2\theta\sin\phi\cos^2\phi\\ &\qquad -\rho^2\sin\phi\cos^2\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta|\\ &= \rho^2\left|\sin^3\phi(\cos^2\theta+\sin^2\theta) + \sin\phi\cos^2\phi(\sin^2\theta+\cos^2\theta)\right|\\ &= \rho^2\left|\sin^2\phi(\sin\phi+\cos^2\phi)\right|\\ &= \rho^2|\sin\phi| = \rho^2\sin\phi.\end{aligned}$$

(We can drop the absolute values around sin because $\sin\phi \geq 0$ when $\phi\in[0,\pi]$).

5.6 Centers of Mass and Other Applications

Now we want to take a bit of time to talk about some of the problems that multiple integrals can solve.

Generally, there are three basic reasons you might want to use an integral, which are all pretty similar:

1. You want to multiply two things together, but they're not constant
2. You want to add up a bunch of numbers
3. You want to take the average of a bunch of numbers.

5.6.1 Density and mass

One application we've seen already is to compute the mass of an object given its density. In theory, we can compute mass by multiplying density and volume. And if an object has constant density, that works fine. But if the density varies depending on position, we can't just multiply "the density" by the volume.

What we can do is chop our object up into tiny cubes, and pretend the density is constant on each cube. This lets us approximate the mass of each cube, and then add them all up—which is an integral!

Example 5.46. Suppose we have a *lamina* (a two-dimensional object with mass) in the shape of a triangle with vertices at $(0,0),(0,3),(3,0)$ (in meters), and density $\rho(x,y) = xy\text{kg/m}^2$. This region has bounds $0 \leq x \leq 3$ and $0 \leq y \leq 3-x$, so we can compute the mass with the integral

$$\begin{aligned} M &= \int_0^3 \int_0^{3-x} xy \, dy \, dx \\ &= \int_0^3 xy^2/2 \Big|_0^{3-x} dx = \int_0^3 \frac{x}{2}(3-x)^2 \, dx \\ &= \frac{1}{2} \int_0^3 x^3 - 6x^2 + 9x \, dx \\ &= \frac{x^4}{8} - x^3 + \frac{9x^2}{4} \Big|_0^3 = \frac{81}{8} - 27 + \frac{81}{4} = \frac{27}{8}. \end{aligned}$$

Thus the lamina has a mass of $\frac{27}{8}$kg.

Note also that the units make sense here. $\rho(x,y)$ has units kg/m^2. And the units of dy and dx are meters, so $\rho(x,y)\,dx\,dy$ has units $\text{kg/m}^2 \cdot \text{m} \cdot \text{m} = \text{kg}$.

5.6.2 Center of Mass

The center of mass of a two dimensional object is, conceptually, the point it can balance on. It is in some sense the "average" location the region occurs. (If the object has constant density, then the center of mass is the geometric center of the object; we call this the *centroid*.)

If the mass of an object occurs in finitely many points, then the center of mass is the weighted average of their locations, where the weighting is by the mass. So if we have particles of mass m_1, m_2, m_3 at points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, with total mass m, then the x-coordinate of the center of mass of the system is

$$\overline{x} = \frac{1}{m} \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3$$

and the y-coordinate is

$$\overline{y} = \frac{1}{m} \sum_{i=1}^{3} m_i y_i = m_1 y_1 + m_2 y_2 + m_3 y_3$$

As a vocabulary note, we say that each of these $m_i x_i$ or $m_i y_i$ is a *(first) moment* of the mass, and the sum $\sum_{i=1}^{n} m_i x_i$ is the *(first) moment of the system* about the y-axis. Note that this may seem backwards: the sum of the x coordinates is the moment about the y axis, because it tells me where the balance point is *along* the horizontal axis but *relative to* the vertical y-axis.

If we have finitely many point masses, we can just add them up like this. But if we have a solid, continuous object, we need to add up infinitely many points. That's exactly where an integral shines.

To find the moment about the x axis we need to add up, essentially, each bit of mass multiplied by its y coordinate. That is, we want to break our object up into little pieces i, j and compute $y_{i,j} m_{i,j}$. But the mass is (approximately) the density times the area, so each piece has moment $y_{i,j} \rho(x_{i,j}, y_{i,j}) \Delta A$. Then to get the full moment of the system, we add up all these little pieces, so we have

$$M_x \approx \sum_{i,j} y_{i,j} \rho(x_{i,j}, y_{i,j}) \Delta A$$
$$M_x = \int_R y \rho(x, y) \, dA.$$

Similarly we get

$$M_y = \int_R x \rho(x, y) \, dA.$$

Then the center of mass is basically the "average" moment. Where, on average, is the object? So we get

$$\overline{x} = \frac{M_y}{m} = \frac{\int_R x\rho(x,y)\,dA}{\int_R \rho(x,y)\,dA}$$
$$\overline{y} = \frac{M_x}{m} = \frac{\int_R y\rho(x,y)\,dA}{\int_R \rho(x,y)\,dA}.$$

Example 5.47. Let's go back to our lamina from example 5.46. We want to find the center of mass, which first means we need to find the moments. We compute

$$\begin{aligned}
M_x &= \int_R y\rho(x,y)\,dA = \int_0^3 \int_0^{3-x} xy^2\,dy\,dx \\
&= \int_0^3 \frac{x}{3}y^3\Big|_0^{3-x}\,dx \\
&= \int_0^3 9x - 9x^2 + 3x^3 - x^4/3\,dx \\
&= \frac{9}{2}x^2 - 3x^3 + \frac{3}{4}x^4 - \frac{1}{15}x^5\Big|_0^3 \\
&= \frac{81}{20} \\
M_y &= \int_R x\rho(x,y)\,dA = \int_0^3 \int_0^{3-x} x^2y\,dy\,dx \\
&= \int_0^3 \frac{x^2}{2}y^2\Big|_0^{3-x}\,dx \\
&= \int_0^3 \frac{1}{2}x^4 - 3x^3 + \frac{9}{2}x^2\,dx \\
&= \frac{1}{10}x^5 - \frac{3}{4}x^4 + \frac{3}{2}x^3\Big|_0^3 \\
&= \frac{81}{20}.
\end{aligned}$$

(We shouldn't be surprised that these answers came out the same! The integrals look completely different, but since the actual object is symmetrical we should get the same answer both ways.)

Then we have

$$\overline{x} = \frac{M_y}{m} = \frac{81/20}{27/8} = \frac{6}{5}$$
$$\overline{y} = \frac{M_x}{m} = \frac{81/20}{27/8} = \frac{6}{5}.$$

So the center of mass of this triangular lamina is at the point $(6/5, 6/5)$.

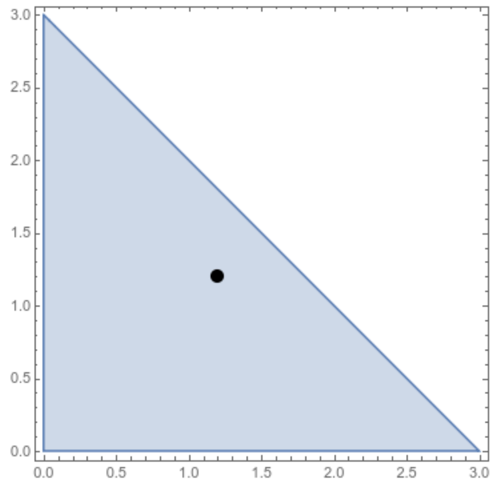
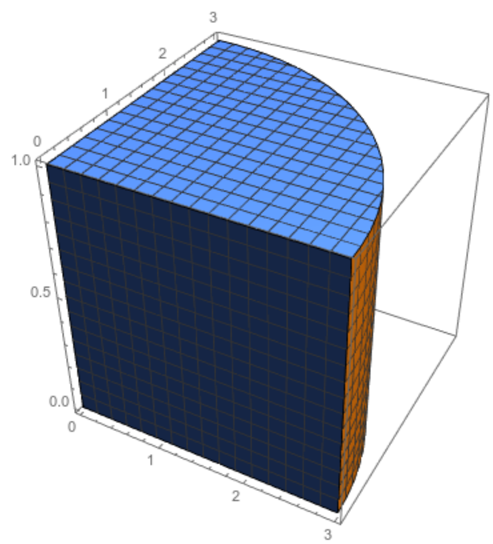


Figure 5.3: The center of mass of a triangular lamina

Example 5.48. Let a solid Q be bounded by $x^2 + y^2 \leq 9, 0 \leq z \leq 1, x \geq 0, y \geq 0$, with density $\rho(x, y, z) = z$.

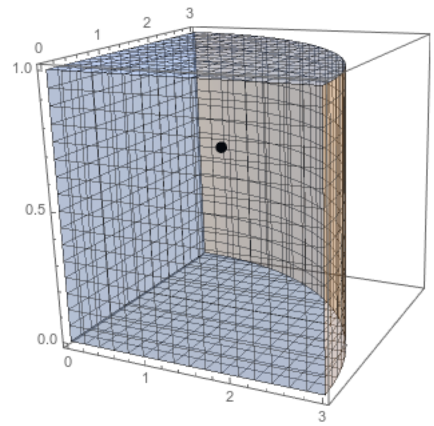


First let's find the mass. We really should use cylindrical coordinates for this; we have

$$
\begin{aligned}
M &= \int_0^1 \int_0^{\pi/2} \int_0^3 z \cdot r \, dr \, d\theta \, dz \\
&= \int_0^1 \int_0^{\pi/2} \frac{r^2}{2} z \Big|_0^3 \, d\theta \, dz \\
&= \int_0^1 \int_0^{\pi/2} \frac{9}{2} z \, d\theta \, dz \\
&= \int_0^1 \frac{9\pi}{4} z \, dz \\
&= \frac{9\pi}{8} z^2 \Big|_0^1 = \frac{9\pi}{8}.
\end{aligned}
$$

Now let's find the center of mass. We need to compute three moments here, about each of the three planes (or along each of the three axes):

$$
\begin{aligned}
M_{xy} &= \int_R z\rho(x,y,z) \, dV \\
&= \int_0^1 \int_0^{\pi/2} \int_0^3 z \cdot z \cdot r \, dr \, d\theta \, dz \\
&= \frac{3\pi}{4} \\
M_{xz} &= \int_R y\rho(x,y,z) \, dV \\
&= \int_0^1 \int_0^{\pi/2} \int_0^3 y \cdot z \cdot r \, dr \, d\theta \, dz \\
&= \int_0^1 \int_0^{\pi/2} \int_0^3 r\sin(\theta) \cdot z \cdot r \, dr \, d\theta \, dz \\
&= \frac{9}{2} \\
M_{yz} &= \int_R x\rho(x,y,z) \, dV \\
&= \int_0^1 \int_0^{\pi/2} \int_0^3 x \cdot z \cdot r \, dr \, d\theta \, dz \\
&= \int_0^1 \int_0^{\pi/2} \int_0^3 r\cos(\theta) \cdot z \cdot r \, dr \, d\theta \, dz \\
&= \frac{9}{2}.
\end{aligned}
$$



Then we can compute

$$\overline{x} = \frac{M_{yz}}{m} = \frac{9/2}{9\pi/8} = \frac{4}{\pi}$$
$$\overline{y} = \frac{M_{xz}}{m} = \frac{9/2}{9\pi/8} = \frac{4}{\pi}$$
$$\overline{z} = \frac{M_{xy}}{m} = \frac{3\pi/4}{9\pi/8} = 2/3.$$

So the center of mass is the point $\left(\frac{4}{\pi}, \frac{4}{\pi}, \frac{2}{3}\right)$.

5.6.3 Moments of Inertia

A related concept is the *moment of inertia* of an object, also known as the *second moment*. The center of mass tells you where the balance point of an object is; the moment of inertia tells you how much torque would be involved in rotating it around an axis.

For a point mass, the formula for the moment of inertia about the x axis is $I_x = my^2$, since y is the distance from the x axis; and the moment of inertia about the y axis is $I_y = mx^2$.

With a non point mass, we need to add up the moments of inertia at every point. Generalizing the argument from section 5.6.2, we get

$$I_x = \int_R y^2 \rho(x,y)\, dA$$
$$I_y = \int_R x^2 \rho(x,y)\, dA.$$

Example 5.49. We can compute the moments of inertia of our lamina from examples 5.46

and 5.47. We have

$$\begin{aligned}
I_x &= \int_0^3 \int_0^{3-x} y^2 \cdot xy\,dy\,dx \\
&= \int_0^3 \frac{x}{4} y^4 \Big|_0^{3-x}\,dx \\
&= \frac{1}{4}\int_0^3 x^5 - 12x^4 + 54x^3 - 108x^2 + 81x\,dx \\
&= \frac{1}{4}\left(\frac{x^6}{6} - \frac{12}{5}x^5 + \frac{27}{2}x^4 - 36x^3 + \frac{81}{2}x^2\right)\Big|_0^3 \\
&= \frac{243}{40} = 6.075 \\
I_y &= \int_0^3 \int_0^{3-x} x^2 \cdot xy\,dy\,dx = 6.075
\end{aligned}$$

Remark 5.50. Sometimes we want the moment of inertia about the origin, called the *polar moment of inertia.* This is just the sum of the x and y moments, and thus

$$I_0 = I_x + I_y = \int_R (x^2+y^2)\rho(x,y)\,dA.$$

Notice that this potentially becomes very nice in polar coordinates: we also have

$$I_0 = \int_R r^2 \rho(r\cos(\theta), r\sin(\theta))\,dA.$$

5.6.4 Probability

Now let's shift gears to a totally different application: probability and expected value. When we have finitely many options, probability is straightforward.

Example 5.51. What is the probability of getting an odd number on a fair six-sided die?

There are three odd numbers, and each one has probability 1/6. So the probability of getting an odd number is $1/6 + 1/6 + 1/6 = 1/2$.

Example 5.52. Now suppose we have an unfair die with the following probabilities:

P(1)	.1	P(2)	.3	P(3)	.15
P(4)	.05	P(5)	.2	P(6)	.2

Now what is the probability of getting an odd number?

We have $P(1) + P(3) + P(5) = .1 + .15 + .2 = .45$.

But often we have infinitely many possibilities. Then we can't add up the odds of each possibility (and in fact, often the odds of any specific outcome are zero: if we pick a number uniformly at random from $[0, 1]$, the probability of picking any specific number is zero). Instead we define a *probability density function* P, and then the probability of getting an outcome in the interval (a, b) is $\int_a^b P(x)\,dx$. The key constraint for a PDF is that the total probability must be 1; so if R is the region containing every possibility, we must have $\int_R P(x)\,dx = 1$.

Example 5.53. The *uniform density* on the interval $[0, 1]$ is given by the probability density function $P(x) = 1$. Then the probability of getting a number in between 1/3 and 2/3 is

$$\int_{1/3}^{2/3} 1\,dx = 1/3$$

as you'd expect.

What if we want to find the probability of landing in a 2-dimensional region?

Example 5.54. Suppose we choose a point from the unit square with the density function

$$P(x, y) = \begin{cases} x + cy^2 & x, y \in [0, 1] \\ 0 & x, y \notin [0, 1] \end{cases}.$$

First we note that only one value of c can make this a probability distribution, because we need the integral over the whole unit square to be 1. So we see

$$\begin{aligned} \int_0^1 \int_0^1 x + cy^2\,dy\,dx &= \int_0^1 xy + cy^3/3 \Big|_0^1 dx \\ &= \int_0^1 x + c/3\,dx \\ &= x^2/2 + cx/3 \Big|_0^1 = 1/2 + c/3. \end{aligned}$$

Since we need this to be equal to 1, we must have $c = 3/2$.

Now let's figure out what the probability of landing in the square $[0, 1/2] \times [0, 1/2]$. We compute

$$\begin{aligned} \int_0^{1/2} \int_0^{1/2} x + \frac{3}{2}y^2\,dy\,dx &= \int_0^{1/2} xy + \frac{1}{2}y^3 \Big|_0^{1/2} dx \\ &= \int_0^{1/2} \frac{x}{2} + \frac{1}{16}\,dx \\ &= \frac{x^2}{4} + \frac{x}{16}\Big|_0^{1/2} = \frac{1}{16} + \frac{1}{32} = \frac{3}{32}. \end{aligned}$$

So the probability of landing in the bottom-left corner of the square is $\frac{3}{32}$.

This can be a purely geometric question, but can also be very concrete, if our two variables represent some concrete physical quantity. Hence we can ask "what is the probability that inflation will be under 5% and unemployment will be under 6%", and we can compute that if we have a probability density function over the two variables representing inflation and unemployment.

5.6.5 Expected Value

We can extend this idea out to compute the expected value of a course of action. In general, the expected value is the value of each possible result, times the probability of each possible result. So if we have finitely many possible outcomes we have

$$E = p_1 v_1 + p_2 v_2 + \cdots + p_n v_n.$$

Example 5.55. Suppose we will roll our weighted die from example 5.52 and get n dollars, where n is the number that comes up on the die.

P(1)	.1	P(2)	.3	P(3)	.15
P(4)	.05	P(5)	.2	P(6)	.2

Then the expected value of rolling the die is

$$.1 \cdot 1 + .3 \cdot 2 + .15 \cdot 3 + .05 \cdot 4 + .2 \cdot 5 + .2 \cdot 6 = 3.55.$$

Again, when we have infinitely many possibilities, we can't simply add them all up individually. Instead we need to compute an integral.

Definition 5.56. The *expected value* of a procedure with probability density function $P(x)$ and value $V(x)$ is

$$E = \int_a^b P(x)V(x)\,dx.$$

If we have a two-variable joint probability, then the expected value is

$$E = \int_R P(x,y)V(x,y)\,dA.$$

One common valuation function V that we care about is just $V(x,y) = x$ or $V(x,y) = y$: that tells us the expected value of x given the probability of getting any particular point (x,y). We write

$$\hat{X} = \int_R xP(x,y)\,dA$$
$$\hat{Y} = \int_R yP(x,y)\,dA.$$

Example 5.57. Suppose we take the unit square with probability density function from example 5.54:

$$P(x,y) = \begin{cases} x + \frac{3}{2}y^2 & x, y \in [0,1] \\ 0 & x, y \notin [0,1] \end{cases}.$$

What is the expected value of x? What is the expected value of y?

$$\begin{aligned}
\hat{X} &= \int_0^1 \int_0^1 x\left(x + \frac{3}{2}y^2\right) dy\,dx \\
&= \int_0^1 \int_0^1 x^2 + \frac{3}{2}xy\,dy\,dx \\
&= \int_0^1 x^2 y + \frac{3}{4}xy^2\Big|_0^1 dx \\
&= \int_0^1 x^2 + \frac{3}{4}x\,dx \\
&= \frac{x^3}{3} + \frac{3}{8}x^2\Big|_0^1 = \frac{1}{3} + \frac{3}{8} = \frac{17}{24}. \\
\hat{Y} &= \int_0^1 \int_0^1 xy + \frac{3}{2}y^3\,dy\,dx \\
&= \int_0^1 \frac{1}{2}xy^2 + \frac{3}{8}y^4\Big|_0^1 dx \\
&= \int_0^1 \frac{1}{2}x + \frac{3}{8}\,dx \\
&= \frac{1}{4}x^2 + \frac{3}{8}x\Big|_0^1 = \frac{1}{4} + \frac{3}{8} = \frac{5}{8}.
\end{aligned}$$

Remark 5.58. If this formula looks familiar, it is: this is the center of mass of a lamina with density given by the PDF we chose!