Math 2233 Fall 2022
Multivariable Calculus Section 12
Mastery Quiz 2
Due Monday, September 12

Everyone should submit both questions on this quiz. I screwed up, and the "vectors" topic that we started last week is in fact a major topic. You get four cracks at it, and will be graded on your best two. Everything is being updated to reflect that.

This week's mastery quiz has two topics. Please do your best on that topic. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material.

Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Vectors
- Secondary Topic 1: Lines and Planes

Name:

Recitation Section:

Major Topic 1: Vectors

(a) Find the orthogonal decomposition of $\vec{v} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ with repsect to $\vec{u} = \vec{i} + 3\vec{j} - 2\vec{k}$.

Solution: First we compute the projection

$$\begin{aligned}\text{Proj}_{\vec{u}}\,\vec{v} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\\ &= \frac{-17}{14}(\vec{i} + 3\vec{j} - 2\vec{k})\\ &= \frac{-17}{14}\vec{i} - \frac{51}{14}\vec{j} + \frac{34}{14}\vec{k}\end{aligned}$$

Now we still need the perpendicular component, but this is a straightforward subtraction:

$$\begin{aligned}\vec{v}_\perp &= \vec{v} - \text{Proj}_{\vec{u}}\,\vec{v}\\ &= 2\vec{i} - 3\vec{j} + 5\vec{k} - \left(\frac{-17}{14}\vec{i} - \frac{51}{14}\vec{j} + \frac{34}{14}\vec{k}\right)\\ &= \frac{45}{14}\vec{i} + \frac{9}{14}\vec{j} + \frac{36}{14}\vec{k}.\end{aligned}$$

(b) Find the area of the **triangle** with vertices $(0, 0, 0), (5, 3, 1), (7, 2, 2)$.

Solution: This triangle is spanned by the vectors $5\vec{i} + 3\vec{j} + \vec{k}$ and $7\vec{i} + 2\vec{j} + 2\vec{k}$, so we compute

$$\begin{aligned}\left(5\vec{i} + 3\vec{j} + \vec{k}\right) \times \left(7\vec{i} + 2\vec{j} + 2\vec{k}\right) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 1 \\ 7 & 2 & 2 \end{vmatrix}\\ &= 6\vec{i} + 7\vec{j} + 10\vec{k} - 21\vec{k} - 2\vec{i} - 10\vec{j}\\ &= 4\vec{i} - 3\vec{j} - 11\vec{k}.\end{aligned}$$

Thus the area of the triangle is $\frac{1}{2}\|4\vec{i} - 3\vec{j} - 11\vec{k}\| = \frac{1}{2}\sqrt{16 + 9 + 121} = \frac{1}{2}\sqrt{146} = \sqrt{73/2}$.

(c) Find $\cos\theta$ where θ is the angle between $\vec{u} = -2\vec{i} + 4\vec{j} + \vec{k}$ and $\vec{v} = 3\vec{i} + \vec{j} + 4\vec{k}$.

Solution:

$$\begin{aligned}\cos\theta &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\\ &= \frac{-6 + 4 + 4}{\sqrt{21}\sqrt{26}}\\ &= \frac{2}{\sqrt{546}}.\end{aligned}$$

Secondary Topic 1: Lines and Planes

(a) Find an equation for the plane that passes through the points $(0, -1, 0)$, $(7, 3, -1)$, and $(4, 2, -2)$

Solution: We can find two vectors in the plane, e.g. $\vec{u} = 7\vec{i} + 4\ vecj - \vec{k}$ and $\vec{v} = 4\vec{i} + 3\vec{j} - 2\vec{k}$. Then teh cross product is

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 7 & 4 & -1 \\ 4 & 3 & -2 \end{vmatrix} = -8\vec{i} - 4\vec{j} + 21\vec{k} - \left(16\vec{k} - 3\vec{i} - 14\vec{j}\right)$$
$$= -5\vec{i} + 10\vec{j} + 5\vec{k}$$

and thus we get the equation

$$-5(x-0) + 10(y+1) + 5(z-0) = 0.$$

(b) Find an equation for the plane perpendicular to $\vec{n} = 3\vec{i} - 2\vec{j} + 7\vec{k}$ that passes through the point $(2, 2, 2)$.

Solution:

$$3(x-2) - 2(y-2) + 7(z-2) = 0.$$

(c) Find a vector perpendicular to the plane given by the equation

$$-2(x-3) + 6(y+1) + 4(z+8) = 0$$

Solution: A normal vector is $-2\vec{i} + 6\vec{j} + 4\vec{k}$.