Math 2233: Multivariable Calculus
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Recitation 2

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Problem 1. Let $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{v} = 2\vec{i} + \vec{j} - \vec{k}$, and $\vec{w} = 3\vec{i} + 2\vec{k}$.

(a) Compute $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$. What do you notice?

(b) We already computed the vector $\vec{u} \times \vec{v}$; now cross that with $\vec{w}$ to get $(\vec{u} \times \vec{v}) \times \vec{w}$.

(c) Do the same vectors, but in a different order. Compute $\vec{v} \times \vec{w}$, and then use that to compute $\vec{u} \times (\vec{v} \times \vec{w})$. What do you notice here?

(d) Can you come up with a geometric argument for why you should expect the result from part (c)? (Try thinking about the case where $\vec{u} = \vec{i}, \vec{v} = \vec{w} = \vec{j}$.)

In class, we talked about the relationship between the equation for a plane and its geometry, encoded in the idea of a normal vector.

If $P = (x_0, y_0, z_0)$ is a point on the plane, then the plane consists of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is perpendicular to $\vec{n}$. Since

$$\overrightarrow{PQ} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k},$$

we see that the plane is the set of points satisfying $\vec{n} \cdot \overrightarrow{PQ} = 0$. If we take $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$, this is

$$\vec{n} \cdot \overrightarrow{PQ} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

So every plane will have an equation that looks like this.

But we can combine this fact with projections to think about the relationships between points and planes.

Problem 2. Let's think about the plane $2(x-1)+1(y-2)-2(z+1)=0$.

(a) Can you find a point on this plane? Can you find a vector perpendicular to the plane?

(b) We want to find the *distance* between the point $Q=(2,3,2)$ and this plane. Can you find a vector from a point in the plane to Q?

(c) We want the *shortest* line from Q to the plane, so we need one that's *perpendicular* to the plane. (Try sketching this!) Do we know a vector that has to be parallel to the vector you just sketched?

(d) We want to combine the two previous answers. We have a vector from the plane to Q, and we know the direction of the shortest vector. If we project the vector from (2) onto the perpendicular vector $\vec{n}$ from (3), what will that look like? Try to sketch a picture. Convince yourself that this projection is a line through Q that's perpendicular to the plane.

(e) Compute this projection.

(f) What is the length of this vector? How far is the point from the plane?

In high school geometry, we characterize a plane by giving three points (not all on one line). We' like to find a way to take in three points and get the equation for a plane. There are two different approaches we can take here.

The most efficient involves the vector operations we've been discussing.

Problem 3. Let's find an equation for the plane containing the three points $(1,4,2),(5,1,1),(-2,1,7)$.

(a) We'd like to turn this into a problem about vector operations, so we need to find some vectors. Can we find vectors that are "in", or more precisely parallel to, this plane?

(b) To find the equation for a plane we need a vector perpendicular to the plane. So we want a vector perpendicular to the vectors from part (a). Do we have a tool that can do that?

(c) Use the result from part (b) to find an equation for the plane.

Problem 4. We want to find the area of the triangle with vertices at the points $(-2,1,3),(4,-1,1)$, and $(1,2,-2)$.

In class we talked about finding the area of a parallelogram. Can we adapt that idea here?