Math 2233: Multivariable Calculus
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Recitation 2

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Problem 1. Let $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$, $\vec{v} = 2\vec{i} + \vec{j} - \vec{k}$, and $\vec{w} = 3\vec{i} + 2\vec{k}$.

(a) Compute $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$. What do you notice?

(b) We already computed the vector $\vec{u} \times \vec{v}$; now cross that with $\vec{w}$ to get $(\vec{u} \times \vec{v}) \times \vec{w}$.

(c) Do the same vectors, but in a different order. Compute $\vec{v} \times \vec{w}$, and then use that to compute $\vec{u} \times (\vec{v} \times \vec{w})$. What do you notice here?

(d) Can you come up with a geometric argument for why you should expect the result from part (c)? (Try thinking about the case where $\vec{u} = \vec{i}, \vec{v} = \vec{w} = \vec{j}$.)

Solution:

(a)

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 1 & -1 \end{vmatrix} = -2\vec{i} + 6\vec{j} + \vec{k} - \left(4\vec{k} + 3\vec{i} - \vec{j}\right) \\
&= -5\vec{i} + 7\vec{j} - 3\vec{k} \\
\vec{v} \times \vec{u} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 3\vec{i} - \vec{j} + 4\vec{k} - \left(\vec{k} - 2\vec{i} + 6\vec{j}\right) \\
&= 5\vec{i} - 7\vec{j} + 3\vec{k}.
\end{aligned}$$

These two results are opposite. This is an illustration of the result from class that the cross product is *anticommutative*: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

(b)

$$\begin{aligned}(\vec{u} \times \vec{v}) \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 7 & -3 \\ 3 & 0 & 2 \end{vmatrix} = 14\vec{i} - 9\vec{j} + 0\vec{k} - \left(21\vec{k} + 0\vec{i} - 10\vec{j}\right) \\ &= 14\vec{i} + \vec{j} - 21\vec{k}.\end{aligned}$$

(c)

$$\begin{aligned}\vec{v} \times \vec{w} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 3 & 0 & 2 \end{vmatrix} = 2\vec{i} - 3\vec{j} + 0\vec{k} - \left(3\vec{k} + 0\vec{i} + 4\vec{j}\right) = 2\vec{i} - 7\vec{j} - 3\vec{k} \\ \vec{u} \times (\vec{v} \times \vec{w}) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & -7 & -3 \end{vmatrix} = -6\vec{i} + 6\vec{j} - 7\vec{k} - \left(4\vec{k} - 21\vec{i} - 3\vec{j}\right) \\ &= 15\vec{i} + 9\vec{j} - 11\vec{k}.\end{aligned}$$

Here we see that the cross product is also not *associative*: $\vec{u} \times (\vec{v} \times vecw) \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

(d) When we think geometrically, it's hard to see why we'd expect the cross product to be associative. It really matters in what order we take perpendicular vectors.

This is maybe easier to see if we take $\vec{u} = \vec{i}, \vec{v} = \vec{w} = \vec{j}$. We know that $\vec{i} \times (\vec{j} \times \vec{j}) = \vec{i} \times \vec{0} = \vec{0}$. (Think about the geometry here!) But $(\vec{i} \times \vec{j}) \times \vec{j} = \vec{k} \times \vec{j} = -\vec{i}$.

Remark 0.1. The cross product does have a somewhat stranger property, though. It satisfies the following relationship, called the *Jacobi Identity*:

$$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \vec{0}.$$

This makes the cross product into something called a *Lie product*, and $\mathbb{R}^3$ into a *Lie Algebra*. These are important for encoding infinitesimal symmetries, and thus are important for quantum mechanics and particle physics.

In class, we talked about the relationship between the equation for a plane and its geometry, encoded in the idea of a normal vector.

If $P = (x_0, y_0, z_0)$ is a point on the plane, then the plane consists of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is perpendicular to $\vec{n}$. Since

$$\overrightarrow{PQ} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k},$$

we see that the plane is the set of points satisfying $\vec{n} \cdot \overrightarrow{PQ} = 0$. If we take $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$, this is

$$\vec{n} \cdot \overrightarrow{PQ} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

So every plane will have an equation that looks like this.

But we can combine this fact with projections to think about the relationships between points and planes.

Problem 2. Let's think about the plane $2(x - 1) + 1(y - 2) - 2(z + 1) = 0$.

(a) Can you find a point on this plane? Can you find a vector perpendicular to the plane?

(b) We want to find the *distance* between the point $Q = (2, 3, 2)$ and this plane. Can you find a vector from a point in the plane to Q?

(c) We want the *shortest* line from Q to the plane, so we need one that's *perpendicular* to the plane. (Try sketching this!) Do we know a vector that has to be parallel to the vector you just sketched?

(d) We want to combine the two previous answers. We have a vector from the plane to Q, and we know the direction of the shortest vector. If we project the vector from (2) onto the perpendicular vector $\vec{n}$ from (3), what will that look like? Try to sketch a picture. Convince yourself that this projection is a line through Q that's perpendicular to the plane.

(e) Compute this projection.

(f) What is the length of this vector? How far is the point from the plane?

Solution:

(a) The point $(1, 2, -1)$ has to be on the plane. By the logic above, the vector $\vec{n} = 2\vec{i}+\vec{j}-2\vec{k}$ is perpendicular to the plane.

(b) We know that $(1, 2, -1)$ is in the plane, so the vector $\vec{i} + \vec{j} + 3\vec{k}$ connects this point to Q.

(c) $\vec{n}$ is perpendicular to the plane, so points in the direction we want.

(d)

(e)

$$\vec{v}_{\text{parallel}} = \frac{\vec{v}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{2+1-6}{4+1+4} = \frac{-3}{9}(2\vec{i}+\vec{j}-2\vec{k}) = \frac{-2}{3}\vec{i} - \frac{1}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

(f)

$$\|\vec{v}_{\text{parallel}}\| = \sqrt{4/9+1/9+4/9} = 1.$$

So the point is exactly one unit away from the plane.

In high school geometry, we characterize a plane by giving three points (not all on one line). We' like to find a way to take in three points and get the equation for a plane. There are two different approaches we can take here.

The most efficient involves the vector operations we've been discussing.

Problem 3. Let's find an equation for the plane containing the three points $(1, 4, 2), (5, 1, 1), (-2, 1, 7)$.

(a) We'd like to turn this into a problem about vector operations, so we need to find some vectors. Can we find vectors that are "in", or more precisely parallel to, this plane?

(b) To find the equation for a plane we need a vector perpendicular to the plane. So we want a vector perpendicular to the vectors from part (a). Do we have a tool that can do that?

(c) Use the result from part (b) to find an equation for the plane.

Solution:

(a) We can find the vectors between any pair of points. We see that the vectors from the first point to the other two are $4\vec{i} - 3\vec{j} - \vec{k}$ and $-3\vec{i} - 3\vec{j} + 5\vec{k}$. We could also find the vector between the second and third points, which is $-7\vec{i} + 6\vec{k}$.

We could also of course multiply any of these vectors by a scalar and still get vectors parallel to the plane. Less obviously, we can add them together and get another vector parallel to the plane. (Question for anyone who's taken linear algebra: what is the name for this set of properties in a linear algebra course?)

(b) The cross product gives a vector perpendicular to two vectors. So we can take the cross product of the vectors we found in (a). If we take $\vec{u} = 4\vec{i} - 3\vec{j} - \vec{k}$ and $\vec{v} = -3\vec{i} - 3\vec{j} + 5\vec{k}$, then we compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & -1 \\ -3 & -3 & 5 \end{vmatrix} = (-15 - 3)\vec{i} + (3 - 20)\vec{j} + (-12 - 9)\vec{k} = -18\vec{i} - 17\vec{j} - 21\vec{k}.$$

(c) An equation for the plane is

$$18(x - 1) + 17(y - 4) + 21(z - 2) = 0.$$

Problem 4. We want to find the area of the triangle with vertices at the points $(-2, 1, 3)$, $(4, -1, 1)$, and $(1, 2, -2)$.

In class we talked about finding the area of a parallelogram. Can we adapt that idea here?

Solution: We can find the vectors from one point to the other two. I take the vectors $\vec{u} = 6\vec{i} - 2\vec{j} - 2\vec{k}$ and $3\vec{i} + \vec{j} - 5\vec{k}$, but there are other choices we could make.

These two vectors span a *parallelogram*. By a result from class, the area of this parallelogram is

$$\begin{aligned} \|\vec{u} \times \vec{v}\| &= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & -2 & -2 \\ 3 & 1 & -5 \end{vmatrix} \right\| \\ &= \left\| \left(10\vec{i} - 6\vec{j} + 6\vec{k}\right) - \left(- 6\vec{k} - 2\vec{i} - 30\vec{j}\right) \right\| \\ &= \left| 12\vec{i} + 24\vec{k} + 12\vec{k} \right\| = 12 \left\| \vec{i} + 2\vec{j} + \vec{k} \right\| 12\sqrt{6}. \end{aligned}$$

That's the area of the parallelogram. But the triangle is one half of the parallelogram, so the triangle has area $6\sqrt{6}$.