Math 2233 Fall 2022
Multivariable Calculus Section 12
Mastery Quiz 3
Due Monday, September 19

This week's mastery quiz has three topics. If you have a 4/4 on M1 you don't need to submit it again; if you have a 2/2 on S1 you don't have ot submit it again. Everyone should submit S2.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Vectors
- Secondary Topic 1: Lines and Planes
- Secondary Topic 2: Vector Functions

Name:

Recitation Section:

Major Topic 1: Vectors

(a) Find the orthogonal decomposition of $\vec{v} = 6\vec{i} + 2\vec{j} - 3\vec{k}$ with repsect to $\vec{u} = 2\vec{i} + \vec{j} + 2\vec{k}$.

Solution: First we compute the projection

$$\begin{aligned} \mathrm{Proj}_{\vec{u}}\,\vec{v} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\ &= \frac{8}{9}(2\vec{i} + \vec{j} + 2\vec{k}) \\ &= \frac{16}{9}\vec{i} + \frac{8}{9}\vec{j} + \frac{16}{9}\vec{k} \end{aligned}$$

Now we still need the perpendicular component, but this is a straightforward subtraction:

$$\begin{aligned} \vec{v}_{\perp} &= \vec{v} - \mathrm{Proj}_{\vec{u}}\,\vec{v} \\ &= 6\vec{i} + 2\vec{j} - 3\vec{k} - \left(\frac{16}{9}\vec{i} + \frac{8}{9}\vec{j} + \frac{16}{9}\vec{k}\right) \\ &= \frac{38}{9}\vec{i} + \frac{10}{9}\vec{j} - \frac{43}{9}\vec{k}. \end{aligned}$$

(b) Find the area of the parallelogram with vertices $(0,0,0), (2,2,2), (1,5,5), (3,7,7)$.

Solution: This parallelogram is spanned by the vectors $2\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{i} + 5\vec{j} + 5\vec{k}$, so we compute

$$\begin{aligned} \left(2\vec{i} + 2\vec{j} + 2\vec{k}\right) \times \left(1\vec{i} + 5\vec{j} + 5\vec{k}\right) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 2 \\ 1 & 5 & 5 \end{vmatrix} \\ &= 10\vec{i} + 2\vec{j} + 10\vec{k} - 2\vec{k} - 10\vec{i} - 10\vec{j} \\ &= -8\vec{j} + 8\vec{k}. \end{aligned}$$

Thus the area of the parallelogram is $\| - 8\vec{j} + 8\vec{k}\| = \sqrt{64+64} = \sqrt{128} = 8\sqrt{2}$.

(c) Find $\cos\theta$ where θ is the angle between $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{v} = \vec{i} - \vec{j} + \vec{k}$.

Solution:

$$\begin{aligned} \cos\theta &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\ &= \frac{1-2+3}{\sqrt{14}\sqrt{3}} \\ &= \frac{2}{\sqrt{42}}. \end{aligned}$$

Secondary Topic 1: Lines and Planes

(a) Find an equation for the plane that passes through the points $(-1, 1, 8), (3, 0, -1)$, and $(2, 2, 3)$.

Solution:

$$\begin{aligned} z &= 8 + m(x+1) + n(y-1) \\ -1 &= 8 + m(3+1) + n(0-1) \\ 3 &= 8 + m(2+1) + n(2-1) \end{aligned}$$

which gives us the system of equations

$$\begin{aligned} -9 &= 4m - n \\ -5 &= 3m + n \end{aligned}$$

The first equation implies that $n = 4m + 9$ and thus the second gives us $-5 = 7m + 9$, which implies $m = -2$ and thus $n = 1$ Then the equation of our plane is

$$z = 8 - 2(x+1) + (y-1).$$

Alternatively, we can take the cross product of the vectors $4\vec{i} - \vec{j} - 9\vec{k}$ and $3\vec{i} + \vec{j} - 5\vec{k}$. This gives us

$$\begin{aligned} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -9 \\ 3 & 1 & -5 \end{vmatrix} &= 5\vec{i} - 27\vec{j} + 4\vec{k} - (-3\vec{k} - 9\vec{i} - 20\vec{)} \\ &= 14\vec{i} - 7\vec{j} + 7\vec{k} \end{aligned}$$

and thus we get the equation

$$14(x+1) - 7(y-1) + 7(z-8) = 0.$$

(b) Find an equation for the plane perpendicular to $\vec{n} = \vec{i} + \vec{4}j - \vec{2}k$ that passes through the point $(5, -3, 0)$.

Solution:

$$(x-5) + 4(y-3) - 2(z-0) = 0.$$

(c) Find a vector perpendicular to the plane given by the equation

$$5(x-4) + 3(y+3) - 7(z-2) = 0$$

Solution: A normal vector is $5\vec{i} + 3\vec{j} - 7\vec{k}$.

Secondary Topic 2: Vector Functions

(a) Find a parametric equation for a particle moving in a straight line, starting at $(0,0,0)$ and moving towards $(3,2,1)$.

(b) Suppose another particle follows the path $\vec{r}_2(t) = (t^2, 9-t, t)$. Does this particle's path intersect the path of the particle from part (a)?

(c) Find an equation for the line tangent to the curve $\vec{r}(t) = (3t, \ln(t^2+1), 5t^2+2)$ at the time $t = 3$.

(d) If a particle moves with velocity $\vec{v}(t) = \langle 6t, \cos(\pi t/2)\rangle$ then what is the displacement between times $t = 1$ and $t = 4$?

Solution:

(a) There are many correct answers, but one is $\vec{r}_1(t) = (3t, 2t, t)$.

(b) The paths intersect if and only if

$$3t_1 = t_2^2 \qquad 2t_1 = 9 - t_2 \qquad t_1 = t_2$$

The last equation tells us the times must be the same; then the first equation gives us that $t_2 = 3$ and the second equation also gives us that $t_2 = 3$. Thus they cross paths at $t_1 = t_2 = 3$.

(c) We have $\vec{r}'(t) = (3, \frac{2t}{t^2+1}, 10t)$ and thus $\vec{r}'(3) = (3, 6/10, 30)$. So a parametric equation for the line is

$$T(t) = (9 + 3t, \ln(10) + 6t/10, 47 + 30t).$$

(d) We have

$$\begin{aligned}\vec{r}(t) &= \int_1^4 6t\vec{i} + \cos(\pi t/2)\vec{j}\,dt \\ &= 3t^2\vec{i} + \frac{2}{\pi}\sin(\pi t/2)\vec{j}\Big|_1^4 \\ &= (48-3)\vec{i} + \frac{2}{\pi}(\sin(2\pi) - \sin(\pi/2))\vec{j} = 45\vec{i} - \frac{2}{\pi}\vec{j}.\end{aligned}$$