Math 2233: Multivariable Calculus
George Washington University   Fall 2022
Recitation 3

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Problem 1. We want to think about how changing a function affects the graph of the function. As a toy example, we'll consider the function $f(x, y) = x^2 + y^2$. In class we saw this looked sort of like a bowl opening upwards.

(a) What should the graph of $f(x, y) + 5 = x^2 + y^2 + 5$ look like?

(b) What should the graph of $f(x - 1, y) = (x - 1)^2 + y^2$ look like?

(c) What should the graph of $f(x, y + 2) = x^2 + (y + 2)^2$ look like?

(d) What should the graph of $2f(x, y) = 2x^2 + 2y^2$ look like?

(e) What should the graph of $f(2x, y) = (2x)^2 + y^2$ look like?

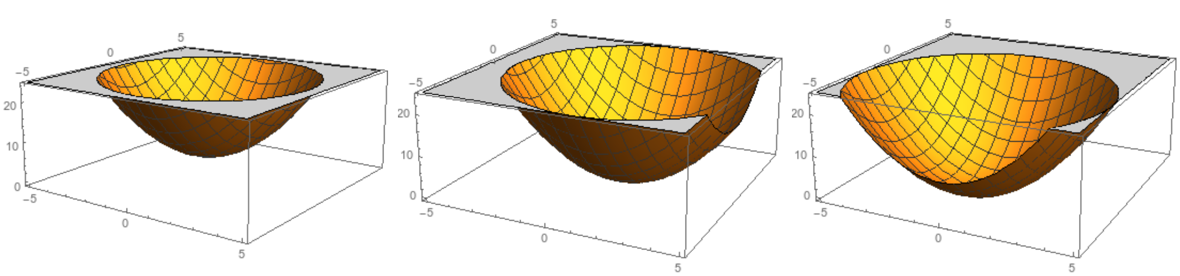


Figure 0.1: The graphs of $f(x, y) + 5$, $f(x - 1, y)$, and $f(x, y + 2)$

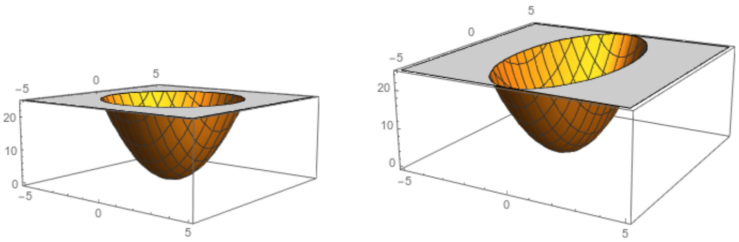


Figure 0.2: The graphs of $2f(x,y)$ and $f(2x,y)$

If $f(x,y)$ is a function of two variables, then we can get a function of one-variable by fixing $x = c$ and considering the function $f(c,y)$. This function is called a *cross-section* of f with x fixed. The graph of this cross-section is the curve given by intersecting the plane $x = c$ with the graph of $f(x,y)$.

Similarly, the function of one variable given by $f(x,c)$ is a cross-section of f with y fixed. The graph of this function is the curve given by intersecting the plane $y = c$ with the graph of $f(x,y)$.

Problem 2. Let's use cross sections to figure out what the function $f(x,y) = x^2 - y^2$ looks like.

(a) Write out the formulas for the single-variable functions $f(x,-2)$, $f(x,-1)$, $f(x,0)$, $f(x,1)$, $f(x,2)$, and $f(x,3)$.

(b) Sketch the graphs of each of the single-variable functions from part (a). Imagine these all stacked up; what do you think the graph of the whole function looks like?

(c) Now write out the formulas for $f(-2,y)$, $f(-1,y)$, $f(0,y)$, $f(1,y)$, $f(2,y)$, and $f(3,y)$.

(d) Sketch the graphs of each of the single-variable functions in part (c). Based on these, what do you think the graph of the whole function looks like?

(e) Are your answers in part (b) and (d) compatible? What is the graph of this function?

Solution:

(a)

$$f(x,-2) = x^2 - 4 \qquad f(x,-1) = x^2 - 1$$
$$f(x,0) = x^2 \qquad f(x,1) = x^2 - 1$$
$$f(x,2) = x^2 - 4 \qquad f(x,3) = x^2 - 9$$

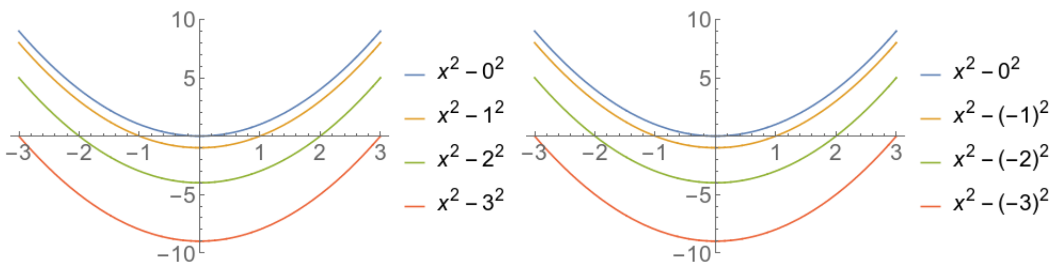


Figure 0.3: Cross sections of $x^2 - y^2$ holding y constant

(b)

(c)

$$f(-2,y) = 4 - y^2 \qquad f(-1,y) = 1 - y^2$$
$$f(0,y) = -y^2 \qquad f(1,y) = 1 - y^2$$
$$f(2,y) = 4 - y^2 \qquad f(3,y) = 9 - y^2$$

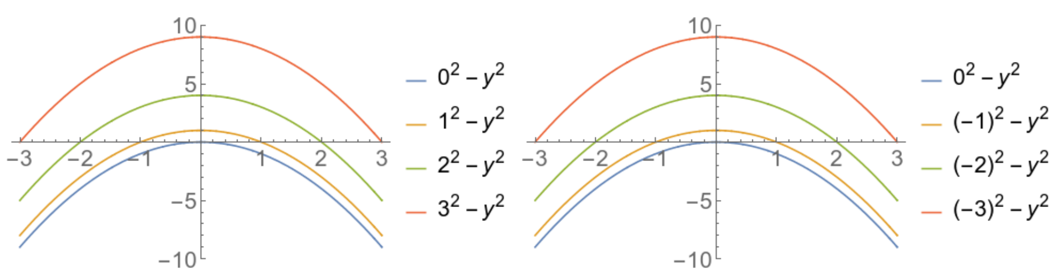


Figure 0.4: Cross sections of $x^2 - y^2$ holding x constant

(d)

(e)

Problem 3. Let's use cross sections to figure out what the function $g(x,y) = xs + \sin(y)$ looks like.

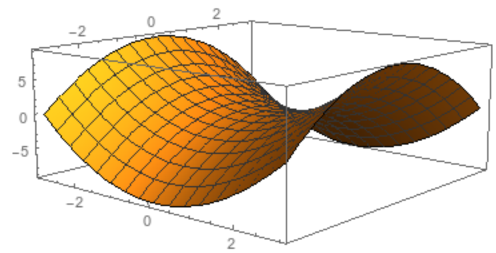


Figure 0.5: The graph of $x^2 - y^2$

(a) Compute $g(-2,y), g(-1,y), g(0,y), g(1,y), g(2,y)$. Sketch the graphs. What does this make you think the graph of g looks like?

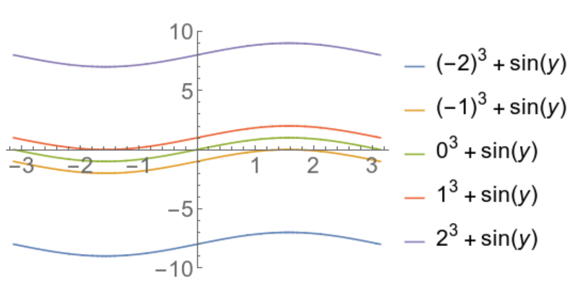
(b) We want to compute cross sections holding y constant, but it doesn't make sense to plug in $-2,-1,0,1,2$ here. What numbers should we plug in? Plug them in and sketch some graphs. What do you think the graph of g looks like?

(c) Are your answers from parts (a) and (b) compatible?

Solution:

(a)

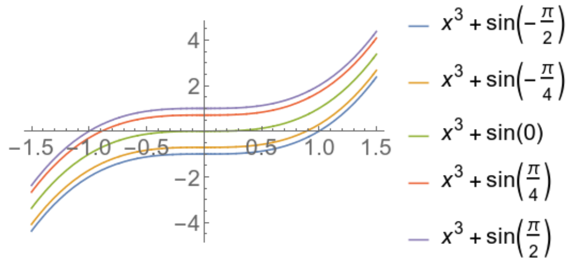
$$g(-2,y) = -8 + \sin(y) \qquad g(-1,y) = -1 + \sin(y)$$
$$g(0,y) = \sin(y) \qquad g(1,y) = 1 + \sin(y) \qquad g(2,y) = 8 + \sin(y)$$



(b) We have to plug these numbers into sin, so we should try things like $-\pi/2, -\pi/4, 0, \pi/4, \pi/2$.

Then we get

$$g(x, -\pi/2) = x^3 - 1 \qquad g(x, -\pi/4) = x^3 - \sqrt{2}/2$$
$$g(x, 0) = x^3 \qquad g(x, \pi/4) = x^3 + \sqrt{2}/2 \qquad g(x, \pi/2) = x^3 + 1$$



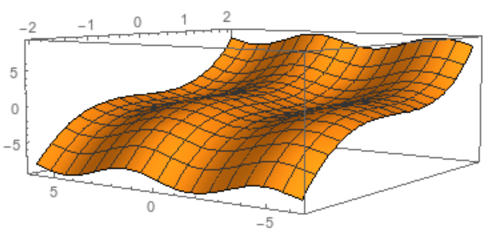


Figure 0.6: The graph of $x^3 + \sin(y)$

(c)

There's also another approach we can take to understanding the graphs of these 2-variable functions. So far we've been fixing one input variable, and graphing the output as the other input varies. But we can also fix the *output* and see what happens.

Definition 0.1. If $f(x, y)$ is a function of two variables, then the *level set* of f at level c is the set of all points (x, y) such that $f(x, y) = c$.

A *contour diagram* for f is a graph with several level sets for f at different levels.

Importantly, the level set is not a function, and doesn't need to pass any vertical line tests or anything similar.

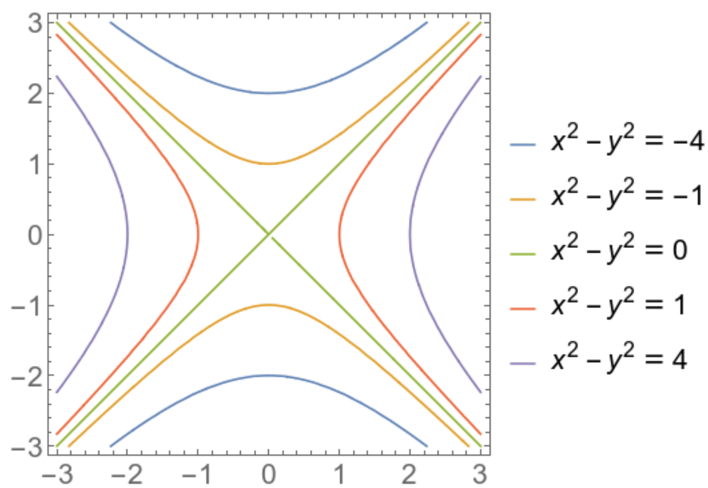
Problem 4. Let's build contour diagrams for the first function we looked at, $f(x, y) = x^2 - y^2$.

(a) Let's start with $f(x, y) = 0$, so we're solving $x^2 - y^2 = 0$. What does this graph look like?

(b) Now let's take some negative-valued level sets. What do the graphs of $f(x, y) = -1$ and $f(x, y) = -4$ look like?

(c) What about $f(x, y) = 1$ and $f(x, y) = 4$?

(d) Draw all these curves on one graph. This is the contour plot. Does it match with the graph we came up with earlier?



Problem 5. Let's look at some other functions we might want to understand.

(a) Let $f(x, y) = 25 - x^2 - y^2$. Compute the level sets for $c = 5, 10, 15, 20, 24$. Draw the contour diagram.

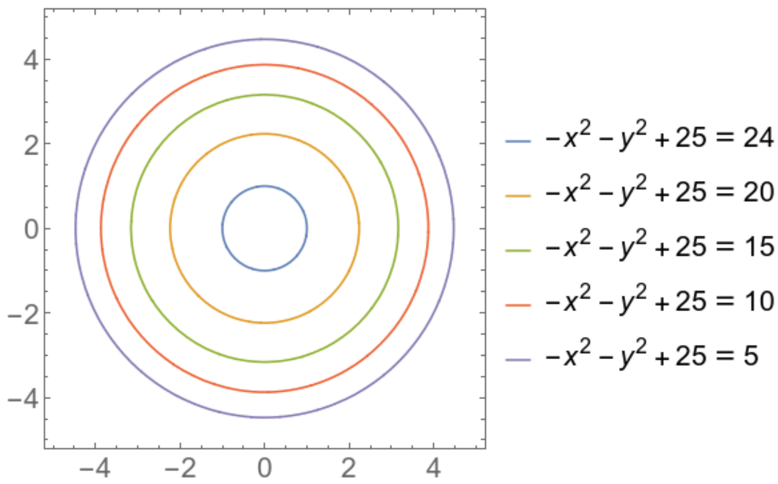
(b) Let $g(x, y) = \sqrt{x^2 + y^2}$. Compute the level sets for $c = 1, 2, 3, 4, 5$. Draw the contour diagram.

(c) Compare these two contour diagrams. How are they different?

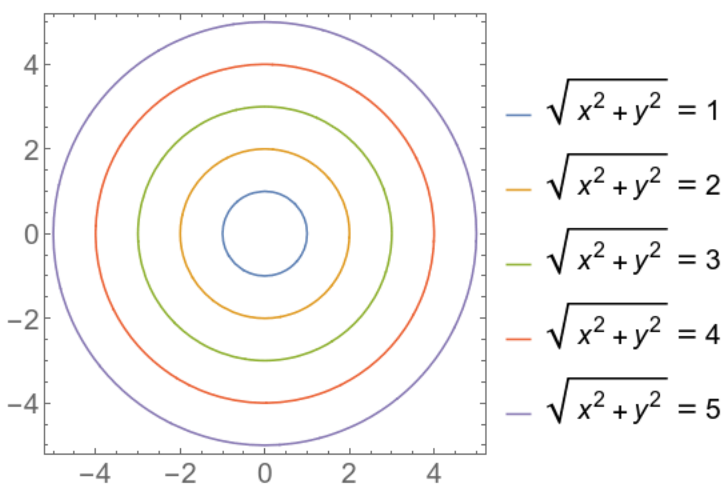
(d) What do these two graphs look like?

Solution:

(a)



(b)



(c) The curves are in roughly the same places. Not exactly, though; if you're careful, the curves for g are spaced evenly but the curves for f are not.

But more importantly, if we *label* the curves they're clearly different. The curves for f increase towards the center, while the curves for g increase away from the center.

(d)

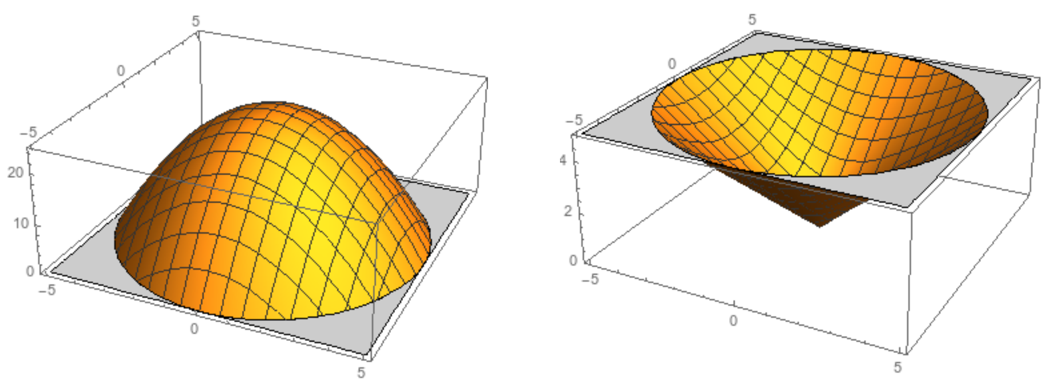


Figure 0.7: Left: $f(x, y) = 25 - x^2 - y^2$. Right: $g(x, y) = \sqrt{x^2 + y^2}$.