Math 1232 Practice Final Solutions

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(a) These are the instructions you will see on the real test, next week. I include them here so you know what to expect.

(b) You will have 120 minutes for this test.

(c) You are not allowed to consult books or notes during the test, but you may use a one-page, two-sided, handwritten cheat sheet you have made for yourself ahead of time.

(d) You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.

(e) This test has 14 questions, over 9 pages. **You should not answer all the questions.**

   (a) The first four problems are one page each, representing the four major topics. You should do all four of them, and they are worth 30?? points each.

   (b) The remaining 10 problems represent topics S1 through S10. You will be graded on your best four, with a few possible bonus points if you also do well on further questions.

   (c) But doing four secondary topics well is much better than doing six poorly.

   (d) If you perform well on a question on this test it will update your mastery scores. Achieving a 27/30 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

Major Topic 1. (a) Compute $\int \frac{x}{\sqrt{4-x^4}}\,dx$.

Solution:

Take $x^2 = 2u$ so $du = x\,dx$. Then

$$\begin{aligned}\int \frac{x}{\sqrt{4-x^4}}\,dx &= \int \frac{1}{\sqrt{4-4u^2}}\,du = \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(x^2/2) + C.\end{aligned}$$

(b) Compute $\int 5^{3x}\,dx$.

Solution: Take $u = 3x$ so $du = 3\,dx$ and we get

$$\begin{aligned}\int 5^{3x}\,dx &= \frac{1}{3}\int 5^u\,du = \frac{1}{3\ln 5}5^u + C \\ &= \frac{1}{3\ln 5}5^{3x} + C.\end{aligned}$$

(c) Write a tangent line to the curve $y^2 = x^{x\cos(x)}$ at the point $(\pi/2, -1)$.

Solution: Implicit differentiation gives us

$$\begin{aligned}2\ln(y) &= x\cos(x)\ln(x) \\ \frac{2y'}{y} &= \cos(x)\ln(x) - x\sin(x)\ln(x) + \cos(x) \\ y' &= \frac{1}{2}\left(\cos(x)\ln(x) - x\sin(x)\ln(x) + \cos(x)\right)y.\end{aligned}$$

When $x = \pi/2, y = -1$, this gives us

$$\begin{aligned}y' &= \frac{1}{2}\left(0\ln(\pi/2) - \pi/2\cdot 1\cdot\ln(\pi/2) + 0\right)(-1) = \frac{1}{2}\left(\pi/2\ln(\pi/2)\right) \\ &= \frac{\pi(\ln(\pi)-\ln(2))}{4}\end{aligned}$$

and thus the tangent line has equation

$$y = \frac{\pi(\ln(\pi)-\ln(2))}{4}(x-\pi/2) - 1.$$

Major Topic 2. (a) $\int \sin x\cos 2x\,dx$

Solution: Take $u = \cos 2x$ and $dv = \sin x\,dx$. We get $du = -2\sin 2x\,dx$ and $v = -\cos x\,dx$, and

$$\begin{aligned}\int \sin x\cos 2x\,dx &= -\cos 2x\cos x - \int 2\sin 2x\cos x\,dx \\ &= -\cos 2x\cos x - 2\left(\int \sin 2x\cos x\,dx\right) \\ &= -\cos 2x\cos x - 2\left(\sin x\sin 2x - 2\int \sin x\cos 2x\,dx\right) \\ &= -\cos x\cos 2x - 2\sin x\sin 2x + 4\int \sin x\cos 2x\,dx \\ -3\int \sin x\cos 2x\,dx &= -\cos x\cos 2x - 2\sin x\sin 2x \\ \int \sin x\cos 2x\,dx &= \frac{1}{3}\left(\cos x\cos 2x + 2\sin x\sin 2x\right)\end{aligned}$$

(b) $\displaystyle\int_{\sqrt{7}}^{2\sqrt{7}} \frac{dx}{x\sqrt{x^2-7}}$

Solution: We see as $\sqrt{x^2-7}$, which should make us think of trigonometric substitution, and in particular $\sqrt{7}\sec\theta = x$. (In the original version of the practice final I posted I had a typo here; see below). We work out $dx = \sqrt{7}\sec\theta\tan\theta\, d\theta$, and the bounds now range from $\sec\theta = 1$ to $\sec\theta = 2$, and thus $\theta = 0$ to $\theta = \pi/3$. Thus

$$\begin{aligned}\int_{\sqrt{7}}^{2\sqrt{7}} \frac{dx}{x\sqrt{x^2-7}} &= \int_0^{\pi/3} \frac{\sqrt{7}\sec\theta\tan\theta\, d\theta}{\sqrt{7}\sec\theta\sqrt{7\sec^2\theta-7}} \\ &= \int_0^{\pi/3} \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\sqrt{7\tan^2\theta}} \\ &= \int_0^{\pi/3} \frac{d\theta}{\sqrt{7}} = \frac{\theta}{\sqrt{7}}\Big|_0^{\pi/3} = \frac{\pi}{3\sqrt{7}}.\end{aligned}$$

(c) $\displaystyle\int \frac{4}{(x^2+1)(x+1)(x-1)}\, dx$

Solution: We have

$$\begin{aligned}\frac{4}{(x^2+1)(x+1)(x-1)} &= \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{x-1} \\ \frac{4}{(x^2+1)(x+1)} &= \frac{(Ax+B)(x-1)}{x^2+1} + \frac{C(x-1)}{x+1} + D \\ \frac{4}{4} &= D \\ \frac{2}{(x^2+1)(x-1)} &= \frac{(Ax+B)(x+1)}{x^2+1} + C + \frac{D(x+1)}{x-1} \\ \frac{4}{-4} &= C\end{aligned}$$

and combining this all gives

$$\begin{aligned}\frac{4}{(x^2+1)(x+1)(x-1)} &= \frac{Ax+B}{x^2+1} + \frac{-1}{x+1} + \frac{1}{x-1} \\ 4 &= (Ax+B)(x+1)(x-1) - (x^2+1)(x-1) + (x^2+1)(x+1) \\ 4 &= B(1)(-1) - (1)(-1) + (1)(1) = -B + 2 \\ -2 &= B \\ 4 &= (Ax-2)(x+1)(x-1) - (x^2+1)(x-1) + (x^2+1)(x+1) \\ 4 &= (2A-2)(3)(1) - (5)(1) + (5)(3) = 6A - 6 - 5 + 15 \\ 0 &= 6A\end{aligned}$$

and thus we have $A = 0, B = -2, C = -1, D = 1$. Thus our integral is

$$\begin{aligned}\int \frac{4}{x^4-1}\, dx &= \int \frac{-2}{x^2+1} + \frac{1}{x-1} - \frac{1}{x+1}\, dx \\ &= -2\arctan(x) + \ln|(x-1)| - \ln|(x+1)| + C.\end{aligned}$$

Major Topic 3. (a) Analyze the convergence of $\displaystyle\sum_{n=2}^{\infty} \frac{3(-1)^n}{n\ln(n)}$. **Solution:** The terms of this series are decreasing and tend to zero, and the series is clearly alternating, so by the alternating series test the series converges. Once we take the absolute value, we have the series $\sum \frac{3}{n\ln n}$. We can't compare this to $\frac{1}{n}$ because that's unhelpful. You may remember from class that this diverges; if not, we compute

$$\begin{aligned}\int_1^{+} \infty \frac{1}{x \ln x}\, dx &= \lim_{t \to +\infty} \int_1^t \frac{1}{x \ln x}\, dx \\ &= \lim_{t \to +\infty} \int_1^{\ln t} \frac{1}{u}\, du \\ &= \lim_{t \to +\infty} \ln u|_1^{\ln t} = \lim_{t \to +\infty} \ln \ln t - \ln 1 = +\infty\end{aligned}$$

so by the integral test the series does not converge absolutely. Thus the series converges conditionally.

(b) Analyze the convergence of $\sum_{n=1}^{\infty} (-1)^n \left(\frac{5n+7}{8n-4}\right)^n$.

Solution: We use the root test. We have

$$\lim_{n \to +\infty} \left| (-1)^n \frac{5n+7}{8n-4} \right| = \frac{5}{8} < 1$$

so the series converges absolutely.

(c) Analyze the convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{n^3+n^2+n+1}{\sqrt{n^9}}$.

Solution: We consider the absolute value of this sequence. The sequence

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n^3+n^2+n+1}{\sqrt{n^9}} \right| = \sum_{n=1}^{\infty} \frac{n^3+n^2+n+1}{\sqrt{n^9}}$$

is positive, so we can use the Limit Comparison Test. We have

$$\lim_{n \to \infty} \frac{(n^3+n^2+n+1)/\sqrt{n^9}}{1/\sqrt{n^3}} = \lim_{n \to \infty} \frac{n^{9/2}+n^{7/2}+n^{5/2}+n^{3/2}}{n^{9/2}} = \lim_{n \to \infty} \frac{1+1/n+1/n^2+1/n^3}{1} = 1.$$

Thus by the Limit Comparison Test, our series converges if and only if $\sum \frac{1}{n^{3/2}}$ converges. But $3/2 > 1$ so this converges, and thus our series converges (absolutely).

Major Topic 4. (a) Find a power series for $\frac{1}{x^3}(e^{2x^3} - 1)$, and write down the first three non-zero terms explicitly.

Solution:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ e^{2x^3} &= \sum_{n=0}^{\infty} \frac{2^n x^{3n}}{n!} && = 1 + 2x^3 + 2x^6 + \frac{4}{3}x^9 + \dots \\ \frac{1}{x^3}(e^{2x^3} - 1) &= \sum_{n=1}^{\infty} \frac{2^n x^{3n-3}}{n!} \left(= \sum_{n=0} \frac{2^{n+1} x^{3n}}{(n+1)!} \right) && = 2 + 2x^3 + \frac{4}{3}x^6 + \dots \end{aligned}$$

(b) Find a power series for $x^2 \arctan(x^2)$ centered at 0.

Solution: We know that $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Thus

$$x^2 \arctan(x^2) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{2n+1}.$$

(c) Find the degree-three Taylor polynomial for $f(x) = \frac{3}{x^3}$ centered at 3.

Solution: First we compute some derivatives:

$$
\begin{aligned}
f(x) &= \frac{3}{x^3} & f(3) &= \frac{1}{9} \\
f'(x) &= -3\frac{3}{x^4} & f'(3) &= -\frac{1}{9} \\
f''(x) &= (3)(4)\frac{3}{x^5} & f''(3) &= \frac{4}{27} \\
f'''(x) &= -(3)(4)(5)\frac{3}{x^6} & f'''(3) &= -\frac{20}{81}
\end{aligned}
$$

Then the Taylor polynomial is

$$T_3(x,3) = \frac{1}{9} - \frac{1}{9}(x-3) + \frac{4/27}{2!}(x-3)^2 + \frac{-20/81}{3!}(x-3)^3.$$

Secondary Topic 1. Let $g(x) = \sqrt[5]{x^9 + x^7 + x + 1}$. Find $(g^{-1})'(1)$.

Solution: We see that $g(0) = 1$, so $g^{-1}(1) = 0$. Then by the Inverse Function Theorem we have

$$
\begin{aligned}
(g^{-1})'(1) &= \frac{1}{g'(g^{-1}(1))} = \frac{1}{g'(0)} \\
g'(x) &= \frac{1}{5}(x^9 + x^7 + x + 1)^{-4/5}(9x^8 + 7x^6 + 1) \\
g'(0) &= \frac{1}{5}(1)(1) = \frac{1}{5}
\end{aligned}
$$

$(g^{-1})'(1) = 5.$

Secondary Topic 2. Compute $\lim_{x\to 0} \frac{e^x - \tan(x) - 1}{x^2}$

Solution:

$$
\begin{aligned}
\lim_{x\to 0} \frac{e^x - \tan(x) - 1}{x^2} &= \lim_{x\to 0} \frac{e^x - \sec^2(x)}{2x} \\
&= \lim_{x\to 0} \frac{e^x - 2\sec^2(x)\tan(x)}{2} = \frac{1}{2}.
\end{aligned}
$$

Secondary Topic 3. Approximate $\int_1^5 3^x\,dx$ with four intervals and Simpson's Rule.

Solution:

$$
\begin{aligned}
\int_1^5 f(x)\,dx &\approx \frac{1}{3}\left(f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)\right) \\
\int_1^5 3^x\,dx &\approx \frac{1}{3}\left(3^1 + 4\cdot 3^2 + 2\cdot 3^3 + 4\cdot 3^4 + 3^5\right) \\
&= 1 + 12 + 18 + 108 + 81 = 220.
\end{aligned}
$$

Secondary Topic 4. $\int_1^{+\infty} \frac{1}{x^2 - 2x}\,dx$

Solution:

$$
\begin{aligned}
\int_1^{+\infty} \frac{1}{x^2 - 2x}\,dx &= \lim_{t\to+\infty} \int_1^t \frac{dx}{x^2 - 2x} \\
&= \lim_{t\to+\infty} \int_1^2 \frac{dx}{x(x-2)} + \int_2^t \frac{dx}{x(x-2)} \\
&= \lim_{r\to 2^-} \int_1^r \frac{dx}{x(x-2)} + \lim_{s\to 2^+} \int_s^3 \frac{dx}{x(x-2)} + \lim_{t\to+\infty} \int_3^t \frac{dx}{x(x-2)}
\end{aligned}
$$

The integral converges if and only if each of these three integrals converges. But let's consider the first one:

$$\begin{aligned}
\lim_{r\to 2^-}\int_1^r \frac{dx}{x(x-2)} &= \lim_{r\to 2^-}\frac{1}{2}\int_1^r \frac{1}{x-2}-\frac{1}{x}\,dx \\
&= \frac{1}{2}\lim_{r\to 2^-}\left(\ln|x-2|-\ln|x|\right)|_1^r \\
&= \frac{1}{2}\lim_{r\to 2^-}\left(\ln(2-r)-\ln(r)-\ln(1)-\ln(1)\right) \\
&= \frac{1}{2}\lim_{r\to 2^-}\left(\ln(2-r)-\ln(r)\right) = -\infty.
\end{aligned}$$

So one of the summands doesn't converge, and thus the integral as a whole diverges.

Secondary Topic 5. Find the area of the surface obtained by rotating the curve $x = 1 + 2y^2$ for $1 \le y \le 2$ about the x-axis.

Solution: Recall we have the formula for surface area $A = \int 2\pi y\,ds$ when we rotate around the x-axis. We will further integrate with respect to y because everything is given as a function of y. We get $x' = 4y$, and thus $ds = \sqrt{1+16y^2}$, so

$$\begin{aligned}
SA &= \int_1^2 2\pi y\sqrt{1+16y^2}\,dy \\
&\quad u = 1+16y^2, du = 32y\,dy \\
&= \int_{17}^{65}\frac{\pi}{16}\sqrt{u}\,du \\
&= \frac{\pi}{16}\frac{2u^{3/2}}{3}\Big|_{17}^{65} = \frac{\pi}{24}\left(65\sqrt{65}-17\sqrt{17}\right).
\end{aligned}$$

Secondary Topic 6. Find the (specific) solution to $y' = x^2y^3$ if $y(0) = 1$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= x^2y^3 \\
\frac{dy}{y^3} &= x^2\,dx \\
\int\frac{dy}{y^3} &= \int x^2\,dx \\
\frac{-1}{2y^2} &= \frac{x^3}{3}+C \\
y^2 &= \frac{-1}{2x^3/3+2C}
\end{aligned}$$

Plugging in $x = 0, y = 1$ gives

$$\begin{aligned}
1 &= \frac{-1}{2C} \\
C &= -1/2 \\
y &= \sqrt{\frac{1}{1/2-2x^3/3}}.
\end{aligned}$$

Secondary Topic 7. Compute $\lim_{n\to\infty}\frac{2^n n!}{(2n)!}$.

Solution: $2^n n! = 2\cdot 4\cdot 6\cdot\cdots\cdot 2n$ and $(2n)! = 1\cdot 2\cdot 3\cdot 4\cdot\cdots\cdot 2n-1\cdot 2n$, so

$$0 \le \frac{2nn!}{(2n)!} = \frac{1}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} \le \frac{1}{2n-1}$$

and since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{2n-1} = 0$, by the Squeeze Theorem $\lim_{n\to\infty} \frac{2^n n!}{(2n!)} = 0$.

Alternatively, we could notice that

$$0 \leq \frac{2^n n!}{(2n!)} = \frac{2^n}{(n+1)(n+2)\dots(2n-1)n} = \frac{2}{n+1}\frac{2}{n+2}\cdots\frac{2}{2n-1}\frac{2}{2n} \leq \frac{1}{n}$$

and since $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0$, and thus by the Squeeze Theorem $\lim_{n\to\infty} \frac{2^n n!}{(2n)!} = 0$.

Alternatively and a bit overpowered-ly, we could consider the series $\sum_{n=1}^{\infty} \frac{2^n n!}{(2n)!}$. Using the ratio test we calculate

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(n+1)!/(2n+2)!}{2^n n!/2n!} \right| = \lim_{n\to\infty} \frac{2(n+1)}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{1}{2n+1} = 0.$$

Thus by the ratio test we see the series converges; and by the divergence test, if the series converges then the sequence of terms must converge to zero.

Secondary Topic 8. Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{(2n)^2+1}$.

Solution: We use the ratio test to find the radius of convergence. We have

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}/((2n+1)^2+1)}{(x-3)^n/((2n)^2+1)} \right| = \lim_{n\to\infty} \frac{|(x-3)|(4n^2+1)}{4n^2+4n+2} = |x-3|.$$

Thus the series converges absolutely when $|x-3| < 1$ and diverges when $|x-3| > 1$, and thus it converges absolutely on $(2,4)$.

When $|x-3| = 1$ we have two points to check. If $x = 4$ then our series is $\sum \frac{1}{(2n)^2+1}$ which converges by the comparison test, since $\frac{1}{(2n)^2+1} < \frac{1}{n^2}$. If $x = 2$ then our series is $\sum \frac{(-1)^n}{(2n)^2+1}$ which converges by the alternating series test. Thus the real interval of convergence is $[2,4]$.

Secondary Topic 9. Use a second-degree Taylor polynomial to approximate $\sqrt[4]{82}$.

Solution: If $g(x) = \sqrt[4]{1+x}$, then by the binomial series we have $g(x) \approx 1 + \frac{x}{4} - \frac{3x^2}{32}$. Then

$$\begin{aligned}
\sqrt[4]{82} = \sqrt[4]{81+1} = 3\sqrt[4]{1+1/81} &\approx 3\left(1 + \frac{1}{81\cdot 4} - \frac{3}{32\cdot 81^2}\right) \\
&= 3 + \frac{1}{27\cdot 4} - \frac{1}{32\cdot 27^2} \\
&= 3 + \frac{1}{108} - \frac{1}{23328} = \frac{70119}{23328} \approx 3.00579.
\end{aligned}$$

Secondary Topic 10. Find an equation for the tangent line to the curve defined by the polar equation $r = 2 + \sin(3\theta)$ at the point $\theta = \pi/4$.

Solution:

We can use our polar equations to parametrize x and y as a function of θ:

$$\begin{aligned}
x &= 2\cos(\theta) + \cos(\theta)\sin(3\theta) \\
y &= 2\sin(\theta) + \sin(\theta)\sin(3\theta) \\
x(\pi/4) &= \sqrt{2} + 1/2 \\
y(\pi/4) &= \sqrt{2} + 1/2.
\end{aligned}$$

Then we can use these parametric equations to find the derivatives of x and y:

$$\begin{aligned}
\frac{dx}{d\theta} &= -2\sin(\theta) - \sin(\theta)\sin(3\theta) + 3\cos(\theta)\cos(3\theta) \\
&= \sqrt{2} + 1/2 - 3/2 = \sqrt{2} - 1 \\
\frac{dy}{d\theta} &= 2\cos(\theta) + \cos(\theta)\sin(3\theta) + 3\sin(\theta)\cos(3\theta) \\
&= -\sqrt{2} - 1/2 - 3/2 = -\sqrt{2} - 2.
\end{aligned}$$

Now we have two choices. First, we can write down a parametric equation for the tangent line. With the slopes we have, this is straightforward. We get

$$\begin{aligned}\vec{r}(t) &= \left(\sqrt{2}+1/2, \sqrt{2}+1/2\right) + t\left(\sqrt{2}-1, -\sqrt{2}-2\right) \\ &= \left(\sqrt{2}+1/2+t\left(\sqrt{2}-1\right), \sqrt{2}+1/2+t\left(-\sqrt{2}-2\right)\right)\end{aligned}$$

Alternatively, we can use our parametric derivatives to find the carterian derivative of y with respect to x:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos(\theta)+\cos(\theta)\sin(3\theta)+3\sin(\theta)\cos(3\theta)}{-2\sin(\theta)-\sin(\theta)\sin(3\theta)+3\cos(\theta)\cos(3\theta)} \\ &= \frac{\sqrt{2}+1/2-3/2}{-\sqrt{2}-1/2-3/2} = \frac{\sqrt{2}-1}{-2-\sqrt{2}}.\end{aligned}$$

And now that we have a slope, we can compute the implicit cartesian equation of this tangent line:

$$y-(\sqrt{2}+1/2) = \frac{1-\sqrt{2}}{2+\sqrt{2}}(x-(\sqrt{2}+1/2)).$$