Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 10
Due Tuesday, April 5

This week's mastery quiz has three topics. This is your first opportunity for M4 and S8. It is the third opportunity for M3, so you may not need to submit that one.

(**Important:** I know I haven't used the word "Taylor Series" in class yet, but the content under M4 is all things we've covered and you should be able to give it a shot.)

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

Analyze the convergence of the following three series. (Specify if they converge absolutely, converge conditionally, or diverge.)

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)+n}{n^2-1}$

Solution: You can't really use the limit comparison test here, at least not easily, because the numerator is a bit over-complicated. But you can use the usual comparison test. We know that $n \leq n + \ln(n)$ and $n^2 - 1 < n^2$, so

$$\frac{\ln(n)+n}{n^2-1} \geq \frac{n}{n^2} = \frac{1}{n}.$$

We know that $\sum \frac{1}{n}$ diverges by the p-series test, so our series diverges by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3n^2+5n+2)^n}{(5n^2-3)^n}$

Solution:

We use the root test. We have

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n(3n^2+5n+2)^n}{(5n^2-3)^n}\right|} = \lim_{n\to\infty} \frac{3n^2+5n+2}{5n^2-3} = \frac{3}{5} < 1$$

So by the root test this converges absolutely..

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{2n+3}$

Solution:

This is an alternating series. Since the terms $\frac{\sqrt{n}}{2m+3}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3}$. You can see this doesn't converge in a couple ways. The integral test isn't super plausible here. You can do a comparison test to $\frac{1}{\sqrt{n}}$: this is larger than $\frac{1}{3\sqrt{n}}$ for large n, and $\frac{1}{3\sqrt{n}}$ diverges. (note: this is *not* larger than $\frac{1}{\sqrt{n}}$!)

It may be easier to use the limit comparison test, though. We have

$$\lim_{n\to\infty} \frac{\sqrt{n}/2n+3}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{n}{2n+3} = 1/2.$$

Since the series $\sum \frac{1}{\sqrt{n}}$ diverges, by the limit comparison test, $\sum \frac{\sqrt{n}}{2n+3}$ diverges, and thus our series does not converge absolutely.

M4: Taylor Series

(a) Write a power series expression for $\frac{2x^2}{4x+1}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\begin{aligned}\frac{1}{1-(-4x)} &= \sum_{n=0}^{\infty}(-4x)^n \\ \frac{2x^2}{1+4x} &= 2x^2\sum_{n=0}^{\infty}(-4)^n x^n \\ &= \sum_{n=0}^{\infty} 2\cdot(-4)^n x^{n+2} \\ \text{(or)} \quad &= \sum_{n=2}^{\infty} 2^{2n-3}(-1)^n x^n.\end{aligned}$$

The radius of convergence is 1/4. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < -4x < 1$ or $-1/4 < x < 1/4$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{2n-1}(-1)^{n+1}x^{n+1}}{2^{2n-3}(-1)^n x^n}\right| = \lim_{n\to\infty} 4|x|$$

and thus it converges when $4|x| < 1$.

(b) If $f(x) = \sum_{n=0}^{\infty}\frac{n+1}{n!+1}x^n$, compute $\int_3^5 f(x)$.

Solution:

$$\begin{aligned}\int f(x) &= \sum_{n=0}^{\infty}\frac{1}{n!+1}x^{n+1} + C \\ \int_3^5 f(x) &= \sum_{n=0}^{\infty}\frac{1}{n!+1}(5^{n+1} - 3^{n+1}).\end{aligned}$$

(c) Write a power series expression for $\ln(x^2)$ centered at 1. What is the radius of convergence?

Solution:

There are a few ways to approach this. One is to observe that $\frac{d}{dx}\ln(x^2) = \frac{2x}{x^2} = \frac{2}{1-(1-x)}$, and by our geometric series power series we have

$$\begin{aligned}
\frac{2}{1-(1-x)} &= 2\sum_0^\infty (1-x)^n \\
&= \sum_0^\infty 2(-1)^n(x-1)^n \\
\ln(x^2) &= \int \sum_0^\infty 2(-1)^n (x-1)^n \, dx \\
&= C + \sum_0^\infty 2(-1)^n \frac{(x-1)^{n+1}}{n+1}
\end{aligned}$$

and plugging 1 in on both sides tells us that $C = 0$. So our power series is

$$\ln(x^2) = \sum_0^\infty 2(-1)^n \frac{(x-1)^{n+1}}{n+1}.$$

(Technically you *do* need to pull out the $(-1)^n$ term for this to be a proper power series centered at 1.)

Alternatively, you could first note that $\ln(x^2) = 2\ln(x)$, so we really just need power series for $\ln(x)$. But we know that

$$\ln(1+y) = \sum_0^\infty (-1)^{n-1} \frac{y^n}{n}.$$

We can take $x = 1 + y$ and thus $x - 1 = y$ and we get

$$\begin{aligned}
\ln(x) &= \sum_0^\infty (-1)^{n-1} \frac{(x-1)^n}{n} \\
\ln(x^2) = 2\ln(x) &= \sum_0^\infty 2(-1)^{n-1} \frac{(x-1)^n}{n}.
\end{aligned}$$

This looks different from our previous answer because it's been reindexed, but it is in fact the same answer.

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=1}^\infty \frac{(2x-5)^n}{n^2}$.

Solution:

We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(2x-5)^{n+1}/(n+1)^2}{(2x-5)^n/n^2}\right| = \lim_{n\to\infty}\left|\frac{(2x-5)n^2}{(n+1)^2}\right|$$
$$= |2x-5|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = |2x-5|.$$

So we need $|2x-5|<1$ or $-1<2x-5<1$, or $4<2x<6$ or $2<x<3$. So the radius is $1/2$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty}\frac{(4-5)^n}{n^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$$

converges by alternating series test

$$\sum_{n=0}^{\infty}\frac{(6-5)^n}{n^2} = \sum_{n=0}^{\infty}\frac{1}{n^2}$$

converges by p-series test

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{n^2x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$.

Solution:

We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}/1\cdot 3\cdot\cdots\cdot(2n+1)}{n^2x^n/1\cdot 3\cdot\cdots\cdot(2n-1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x}{n^2(2n+1)}\right|$$
$$= |x|\lim_{n\to\infty}\frac{(n+1)^2}{n^2(2n+1)} = 0.$$

This is always less than 1, so the series always converges. The radius of convergence is ∞ and the interval of convergence is $(-\infty,\infty)$.