Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 11
Due Tuesday, April 12

This week's mastery quiz has three topics. This is your second opportunity for M4, so everyone should submit that one. It is the final opportunity for M3 and S8.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Taylor Series
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

Analyze the convergence of the following three series. (Specify if they converge absolutely, converge conditionally, or diverge.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3+n}$

Solution:

We use the Ratio test. We have

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}/(n+1)^3+n+1}{(-2)^n/n^3+n} \right| = \lim_{n\to\infty} \frac{2(n^3+n)}{(n+1)^3+n+1}$$
$$= \lim_{n\to\infty} 2 > 1.$$

This limit is greater than 1, so by the ratio test this diverges.

Alternatively, we could note that

$$\lim_{n\to\infty} \frac{(-2)^n}{n^3+n} = \pm\infty,$$

so by the ratio test this diverges. But it's a little tricky to cleanly argue that this goes to infinity; we can't really use L'Hospital's rule without getting the negative sign out of there somehow.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Solution: This clearly converges by the alternating series test, since $\lim_{n\to\infty} \frac{1}{n^2} = 0$, but does it absolutely converge? The ratio test won't work; if we work it out we'll get a limit of 1. But we have

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by the p-series test, so our series converges absolutely. (And thus we don't actually need to check for whether the alternating series test applies.

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2+1}$

Solution: We can work this out with the integral test. We have

$$\int_1^{\infty} xe^{-x^2+1}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x^2+1}\,dx \;\; = \lim_{t\to\infty} \frac{-1}{2} e^{-x^2+1}\Big|_1^t = \lim_{t\to\infty} \frac{1}{2}e^2 - \frac{1}{2}e^{-t^2+1} = e^2/2 < \infty.$$

Since this integral converges, the series must also converge by the integral test.

Alternatively, we could use the ratio test. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)e^{-n^2-2n}}{ne^{-n^2+1}} &= \lim_{n\to\infty}\frac{n+1}{n}\frac{e^{n^2-1}}{e^{n^2+2n}}\\ &= \lim_{n\to\infty}\frac{n+1}{n}\frac{1}{e^{2n+1}} = 0.\end{aligned}$$

Since $0 < 1$, this converges by the Ratio Test.

M4: Taylor Series

(a) Let $f(x) = \cos^2(x)$. Use *the definition of a Taylor series* to find $T_4(x, \pi)$ for this function. (That is, find the terms up through the cubic term.)

Solution:

$$\begin{aligned} f(x) &= \cos^2(x) & f(\pi) &= 1\\ f'(x) &= -2\cos(x)\sin(x) & f'(\pi) &= 0\\ f''(x) &= 2\sin^2(x) - 2\cos^2(x) & f''(\pi) &= -2\\ f'''(x) &= 4\sin(x)\cos(x) + 4\cos(x)\sin(x) & f'''(\pi) &= 0\\ f^{(4)} &= 8\cos^2(x) - 8\sin^2(x) & f^{(4)}(\pi) &= 8.\end{aligned}$$

So we have

$$T_4(x,\pi) = 1 - (x-\pi)^2 + \frac{1}{3}(x-\pi)^4.$$

(b) Using the Taylor series remainder, show that $\sin(x)$ is equal to its Maclaurin series.

Solution: We know that $\sin(x) = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$. We know that $f^{n+1}(x) = \pm\cos(x)$ or $\pm\sin(x)$ so $|f^{n+1}(z)| \leq 1$, and thus

$$\begin{aligned}|R_n(x)| &= \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!}\\ 0 \leq |R_n(x)| &\leq \frac{x^{n+1}}{(n+1)!}\end{aligned}$$

and we know that for any x, $\lim_{n\to\infty} x^{n+1}/(n+1)! = 0$. Thus, by the squeeze theorem, $\lim_{n\to\infty} R_n(x) = 0$ for any x, and thus $\sin(x) = T_{\sin}(x, 0)$ for any x.

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $(1-2x)^{-3}$, and then write down the degree-four polynomial explicitly.

Solution:

We can take this from the binomial series. So we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty}\binom{-3}{n}(-2x)^n = \sum_{n=0}^{\infty}\binom{-3}{n}(-2)^n x^n\\ T_4(x,0) &= 1 + (-2)\frac{-3}{1}x + 4\frac{12}{2}x^2 + (-8)\frac{-60}{6}x^3 + (16)\frac{360}{24}x^4\\ &= 1 + 6x + 24x^2 + 80x^3 + 240x^4\end{aligned}$$

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(5x-3)^n}{\sqrt{n}}$.

Solution:

We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(5x-3)^{n+1}/\sqrt{n+1}}{(5x-3)^n/\sqrt{n}} \right| = \lim_{n\to\infty} \left| \frac{(5x-3)\sqrt{n}}{\sqrt{n+1}} \right|$$
$$= |5x-3| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = |5x-3|.$$

So we need $|5x-3| < 1$ or $-1 < 5x-3 < 1$, or $2 < 5x < 4$ or $2/5 < x < 4/5$. We need to have x in the interval $(3/5 - 1/5, 3/5 + 1/5)$, so the radius is $1/5$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{(2-3)^n}{\sqrt{n}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by alternating series test

$$\sum_{n=0}^{\infty} \frac{(4-3)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by p-series test

Thus the interval of convergence is $[2/5, 4/5)$.

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n}{5^n}(x-3)^n$.

Solution:

We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}/5^{n+1}}{(n)(x-3)^n/5^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{x-3}{5} \right|$$
$$= |x-3|/5 \lim_{n\to\infty} \frac{n+1}{n} = |x-3|/5.$$

So we need $|x-3|/5 < 1$ or $-5 < x-3 < 5$, or $-2 < x < 8$ or $3-5 < x < 3+5$. So the radius is 5.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{n}{5^n} 5^n = \sum_{n=0}^{\infty} n$$

diverges by divergence or p-series test

$$\sum_{n=0}^{\infty} \frac{n}{5^n} (-5)^n = \sum_{n=0}^{\infty} (-1)^n n$$

diverges by divergence test

Thus the interval is $(-2, 8)$.