Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 13
Due Tuesday, April 26

This week's optional mastery quiz has three topics. It is the last opportunity to raise your mastery scores before the final. You should turn the quiz in either on blackboard, or in my office before I leave for the day on Tuesday.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Taylor Series
- Secondary Topic 9: Taylor Series Applications
- Secondary Topic 10: Parametrization

Name:

Recitation Section:

M4: Taylor Series

(a) Using series we already know, write down a formula for the (infinite) Taylor series for $e^{3x} - e^x$, and then write down the degree-three polynomial explicitly.

Solution:

We can take this from the known series for e^x. So we have

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
e^{3x} &= \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n \\
e^{3x} - e^x &= \sum_{n=0}^{\infty} \frac{3^n - 1}{n!} x^n \\
T_3(x, 0) &= 0 + \frac{2}{1}x + \frac{8}{2}x^2 + \frac{26}{6}x^3 \\
&= 2x + 4x^2 + \frac{13}{3}x^3.
\end{aligned}$$

(b) Let $f(x) = \sin(x)$. Use *the definition of a Taylor series* to find $T_3(x, \pi/6)$ (centered at $\pi/6$) for this function. (That is, find the terms up through the degree-three term.)

Solution:

$$\begin{aligned}
f(x) &= \sin(x) & f(\pi/4) &= 1/2 \\
f'(x) &= \cos(x) & f'(\pi/4) &= \sqrt{3}/2 \\
f''(x) &= -\sin(x) & f''(\pi/4) &= -1/2 \\
f'''(x) &= -\cos(x) & f'''(\pi/4) &= -\sqrt{3}/2
\end{aligned}$$

So we have

$$T_3(x, \pi/6) = 1/2 + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{4}(x - \pi/6)^2 - \frac{\sqrt{3}}{12}(x - \pi/6)^3$$

(c) Write a power series expression for $\frac{x}{2+x^2}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\begin{aligned}
\frac{1}{2-x} &= \frac{1}{2}\frac{1}{1 - x/2} = \frac{1}{2}\sum_{n=0}^{\infty}(x/2)^n \\
\frac{1}{2+x^2} &= \frac{1}{2}\frac{1}{1 - (-x^2/2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-x^2/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n} \\
\frac{x}{2+x^2} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+1}.
\end{aligned}$$

The radius of convergence is $\sqrt{2}$. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < x^2/2 < 1$ or $-2 < x^2 < 2$ or $-\sqrt{2} < x < \sqrt{2}$. Or we can use the ratio test:

$$\lim_{n\to\infty} \left| \frac{x^{2n+3}/2^{n+2}}{x^{2n+1}/2^{n+1}} \right| = \lim_{n\to\infty} \frac{|x|^2}{2}$$

and thus it converges when $x^2/2 < 1$.

S9: Applications of Taylor Series

(a) Use a Taylor series to compute $\lim_{x\to 0} \frac{xe^{x^3} - x - x^4}{x^7} =$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{xe^{x^3} - x - x^4}{x^7} &= \lim_{x\to 0} \frac{(x + x^4 + x^7/2 + x^{10}/3! + \dots) - x + x^4}{x^7} \\
&= \lim_{x\to 0} \frac{x^7/2 + x^{10}/3! + \dots}{x^7} \\
&= \lim_{x\to 0} \frac{1}{2} - \frac{x^3}{3!} + \dots = \frac{1}{2}.
\end{aligned}$$

(b) Use a degree-three Taylor polynomial to estimate $(1.1)^{3.1}$.

Solution:

$$\begin{aligned}
(1.1)^{3.1} &\approx 1 + 3.1x + \frac{3.1 \cdot 2.1}{1 \cdot 2}x^2 + \frac{3.1 \cdot 2.1 \cdot 1.1}{1 \cdot 2 \cdot 3}x^3 \\
&= 1 + 3.1x + 3.255x^2 + 1.1935x^3 \\
(1.1)^{3.1} &\approx 1 + 3.1(.1) + 3.255(.1)^2 + 1.1935(.1)^3 = 1 + .31 + .03255 + .0011935 = 1.3437435.
\end{aligned}$$

(c) Use a degree-five Taylor polynomial to estimate $\sin(.3)$.

Solution:

We have

$$\begin{aligned}
\sin(x) &\approx x - x^3/6 + x^5/120 \\
\sin(.3) &\approx .3 - .3^3/6 + .3^5/120 \approx .29552.
\end{aligned}$$

S10: Parametrization

(a) Find the length of the curve parametrized by $x = e^t - t, y = 4e^{t/2}$ for $0 \le t \le 2$.

Solution:

We have $x'(t) = e^t - 1$ and $y'(t) = 2e^{t/2}$, so the arc length is

$$\begin{aligned} L &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_0^2 \sqrt{(e^t-1)^2 + (2e^{t/2})^2}\, dt \\ &= \int_0^2 \sqrt{e^{2t} - 2e^2 + 1 + 4e^t}\, dt = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\, dt \\ &= \int_0^2 e^t + 1\, dt = e^t + t\Big|_0^2 = e^2 + 2 - 1 = e^2 + 1. \end{aligned}$$

(b) Find an equation of the line tangent to the curve $x = \cos^3(t), y = \sin^3(t)$ at the point $(1/8, -3\sqrt{3}/8)$.

Solution:

We have $x'(t) = -3\cos^2(t)\sin(t)$ and $y'(t) = 3\sin^2(t)\cos(t)$. This point happens at $t = -\pi/3$, so we have

$$\begin{aligned} x'(-\pi/3) &= 3\sqrt{3}/8 \\ y'(-\pi/3) &= 9/8 \\ \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{9/8}{3\sqrt{3}/8} = \sqrt{3} \\ y + \frac{3\sqrt{3}}{8} &= \sqrt{3}(x - 1/8). \end{aligned}$$

(c) Find the area inside the cardioid $r = 1 + \cos(\theta)$.

Solution:

We have the polar area formula

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(1 + \cos(\theta))^2\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2} + \cos(\theta) + \frac{1}{2}\cos^2(\theta)\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 1 + 2\cos(\theta) + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta \\ &= \frac{1}{2}\left(\theta + 2\sin(\theta) + \frac{\theta}{2} + \frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} \\ &= \frac{1}{2}\left((2\pi + 0 + \pi + 0) - (0 + 0 + 0 + 0)\right) = \frac{3\pi}{2}. \end{aligned}$$