Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 4
Due Tuesday, February 8

This week's mastery quiz has three topics. Everyone should submit M2. If you have a 2/2 on S2 from last week, you shouldn't submit it. If you have a 4/4 on M1, meaning you got 2/2 on *both* of your first two attempts, you shouldn't submit that.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 2: L'Hospital's rule

Name:

Recitation Section:

M1: Calculus of Transcendental Functions

(a) $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx =$

Solution:

We can factor a 4 out to get $\frac{1}{2}\frac{1}{\sqrt{1-x^2/4}}$. Then we set $u = x/2$, and $du = 1/2\,dx$, and we have

$$\begin{aligned}\int \frac{1}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{2}\frac{2}{\sqrt{1-u^2}}\,du \\ &= \int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \arcsin u + C = \arcsin(x/2) + C.\end{aligned}$$

(b) (Note this is a definite integral)

$\displaystyle\int_0^2 \frac{e^x}{e^x+1}\,dx =$

Solution: We can take $u = e^x$ so $du = e^x\,dx$ and

$$\int_0^2 \frac{e^x}{e^x+1}\,dx = \int_1^{e^2} \frac{1}{u+1}\,du = \ln|u+1|\Big|_1^{e^2} = \ln(e^2+1) - \ln(2).$$

(c) Compute $\displaystyle\frac{d}{dx}\left(\sqrt{x+1}\right)^x$

Solution:

$$\begin{aligned} y &= \sqrt{x+1}^x \\ ln|y| &= x\ln(\sqrt{x+1}) = \frac{1}{2}x\ln(x+1) \\ y'/y &= \frac{1}{2}\left(\ln(x+1) + \frac{x}{x+1}\right) \\ y' &= \frac{1}{2}\sqrt{x+1}^x\left(\ln(x+1) + \frac{x}{x+1}\right)\end{aligned}$$

M2: Advanced Integration Techniques

(a) $\displaystyle\int \sin(2x)\cos(3x)\,dx =$

Solution:

$$\int \sin(2x)\cos(3x)\,dx = \frac{1}{3}\sin(2x)\sin(3x) - \int \frac{2}{3}\cos(2x)\sin(3x)\,dx$$
$$\int \cos(2x)\sin(3x)\,dx = -\frac{1}{3}\cos(2x)\cos(3x) - \int \frac{2}{3}\sin(2x)\cos(3x)$$
$$\int \sin(2x)\cos(3x)\,dx = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}\int \sin(2x)\cos(3x)\,dx$$
$$\frac{5}{9}\int \sin(2x)\cos(3x)\,dx = \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x)\big(+C\big)$$
$$\int \sin(2x)\cos(3x)\,dx = \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.$$

(b) $\int \cos^3(2x)\,dx =$

Solution:

$$\int \cos^3(2x)\,dx = \int \cos(2x)(1-\sin^2(2x))\,dx$$
$$= \int \cos(2x) - \sin^2(2x)\cos(2x)\,dx$$
$$\frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C.$$

(c) $\int \sec^4(3t)\,dt =$

Solution:

We're going to take $u = \tan(3t)$ so that $du = 3\sec^2 3t\,dt$. Then

$$\int \sec^4(3t)\,dt = \int \sec^2(3t)(1+\tan^2(3t))\,dt$$
$$= \int \frac{1}{3}(1+u^2)\,du$$
$$= \frac{u}{3} + \frac{u^3}{9} + C$$
$$= \frac{1}{9}\big(3\tan(3t) + \tan^3(3t)\big) + C.$$

S2: L'Hospital's rule

(a) $\lim_{x\to 0}\left(\frac{e^x+1}{2}\right)^{1/x} =$

Solution:

$$\begin{aligned}
\ln y &= \frac{1}{x}\ln\left(\frac{e^x+1}{2}\right) \\
\lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{\ln\left(\frac{e^x+1}{2}\right)^{\nearrow 0}}{x_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to 0}\frac{2}{e^x+1}\cdot\frac{e^x}{2} = \lim_{x\to 0}\frac{e^x}{e^x+1} = 1/2 \\
\lim_{x\to 0} y &= e^{1/2}.
\end{aligned}$$

(b) $\lim\limits_{x\to+\infty}\dfrac{\arctan(x)}{\arctan(x)+1} =$

Solution: $\lim_{x\to+\infty}\arctan(x) = \pi/2$, so this limit is $\frac{\pi/2}{\pi/2+1} \approx .611$.

Note: you cannot use L'Hospital's rule here! If you tried, you would get

$$\lim_{x\to+\infty}\frac{1/(x^2+1)}{1/(x^2+1)} = \lim_{x\to+\infty} 1 = 1$$

but that is not in fact the limit.

(c) $\lim\limits_{x\to 1}\dfrac{\ln(x)}{\arcsin(x-1)} =$

Solution:

The top and bottom both approach 0, so we can use L'Hospital's Rule:

$$\begin{aligned}
\lim_{x\to 1}\frac{\ln(x)^{\nearrow 0}}{\arcsin(x-1)_{\searrow 0}} &=^{\text{L'H}} \lim_{x\to 1}\frac{1/x}{\frac{1}{\sqrt{1-(x-1)^2}}} \\
&= \lim_{x\to 1}\frac{\sqrt{1-(x-1)^2}}{x} = 1.
\end{aligned}$$