Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 5
Due Tuesday, February 15

This week's mastery quiz has three topics. Everyone should submit M2 and S3. If you have a 4/4 on M1, meaning you got 2/2 on at least two of your first three attempts, you shouldn't submit that.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 3: Numeric Integration

Name:

Recitation Section:

M1: Calculus of Transcendental Functions

(a) $\frac{d}{dx}\frac{1}{\arcsin(x^2)} =$

Solution:

$$\frac{d}{dx}\frac{1}{\arcsin(x^2)} = \frac{-1}{\arcsin(x^2)^2}\frac{1}{\sqrt{1-x^4}} \cdot 2x.$$

(b) $\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\,dx =$

Solution:

We can take $u = \cos(x)$ so that $du = -\sin(x)\,dx$. Then

$$\int \frac{\cos(x)\sin(x)}{1+\cos^2(x)}\,dx = \int \frac{-u}{1+u^2}\,du$$

Then we can set $v = 1+u^2$ so that $dv = 2u\,du$ and we get

$$\begin{aligned}\int \frac{-u}{1+u^2}\,du &= \int \frac{-1}{2}\frac{1}{v}\,dv = \frac{-1}{2}\ln|v| + C \\ &= \frac{-1}{2}\ln|1+u^2| + C = \frac{-1}{2}\ln|1+\cos^2(x)| + C.\end{aligned}$$

(c) $\int \frac{x}{9+x^4}\,dx =$

Solution:

We can factor a 9 out to get $\frac{1}{9}\frac{x}{1+x^4/9}$. Then we set $u = x^2/3$, and $du = 2x/3\,dx$, and we have

$$\begin{aligned}\int \frac{x}{9+x^4}\,dx &= \int \frac{1}{9}\frac{x}{1+u^2}\frac{3}{2x}\,du \\ &= \int \frac{1}{6}\frac{1}{1+u^2}\,du \\ &= \frac{1}{6}\arctan u + C = \frac{1}{6}\arctan(x^2/3) + C.\end{aligned}$$

M2: Advanced Integration Techniques

(a) Compute $\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx =$

Solution:

$$
\begin{aligned}
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1} \\
x^2+x-4 &= A(x+3)(x+1) + B(x+1) + C(x+3)^2 \\
2 &= -2B \Rightarrow B = -1 \\
-4 &= 4C \Rightarrow C = -1 \\
-4 &= 3A + B + 9C = 3A - 1 - 9 \Rightarrow A = 2 \\
\frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1} \\
\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx &= \int \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1}\,dx \\
&= 2\ln|x+3| + \frac{1}{x+3} - \ln|x+1| + C.
\end{aligned}
$$

(b) $\int x^3\sqrt{1-x^2}\,dx =$

Solution:

We're going to set $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$. Then

$$
\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)\,d\theta \\
&= \int \sin^3(\theta)\cos^2(\theta)\,d\theta \\
&= \int \cos^2(\theta)\sin(\theta)(1-\cos^2(\theta))\,d\theta \\
&= \int \cos^2(\theta)\sin(\theta) - \cos^4(\theta)\sin(\theta)\,d\theta.
\end{aligned}
$$

At this point we can take $u = \cos(\theta)$ so that $du = -\sin(\theta)\,d\theta$, and we get

$$
\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx = \int -u^2 + u^4\,du &= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5(\theta)}{5} - \frac{\cos^3(\theta)}{3} + C.
\end{aligned}
$$

But we know that $\sin(\theta) = x$, so $\cos(\theta) = \sqrt{1-x^2}$, and we have any of the following acceptable answers:

$$
\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \frac{\sqrt{1-x^2}^5}{5} - \frac{\sqrt{1-x^2}^3}{3} + C \\
&= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C \\
&= (1-x^2)^{3/2}\left(\frac{1-x^2}{5} - \frac{1}{3}\right) + C.
\end{aligned}
$$

S3: Numeric Integration

(a) Let $f(x) = x^3 + x$. How many intervals do you need with the midpoint rule to approximate $\int_1^2 x^3 + x\,dx$ to within 1/10? Compute that approximation. (Feel free to use a calculator to plug values into f, but show every step.)

Solution:

We have

$$\begin{aligned} f''(x) &= 6x \\ f'(2) &= 12 \\ |E_M| &\leq \frac{12 \cdot 1^3}{24 \cdot n^2} \leq \frac{1}{10} \\ n^2 &\geq 5 \\ n &> 2 \end{aligned}$$

so we need to use at least three intervals. Then the midpoint approximation would be

$$\int_1^2 x^3 + x\,dx \approx \frac{1}{3}f(7/6) + \frac{1}{3}f(9/6) + \frac{1}{3}f(11/6) \approx \frac{1}{3}(2.75 + 4.875 + 8.00) = \frac{1}{3}15.625 \approx 5.21.$$

(Since the true answer is 5.25 this is in fact within our error bound.)

(b) Suppose we have

$$g(0) = 2.4 \qquad g(1) = 3 \qquad g(2) = 2.7 \qquad g(3) = 2.3 \qquad g(4) = 1.9$$

Approximate $\int_0^4 g(x)\,dx$ using the Trapezoid rule and using Simpson's rule.

Solution:

For the trapezoid rule, we have

$$\begin{aligned} T_4 &= 1 \cdot \frac{2.4 + 3}{2} + 1 \cdot \frac{3 + 2.7}{2} + 1 \cdot \frac{2.7 + 2.3}{2} + 1 \cdot \frac{2.3 + 1.9}{2} \\ &= \frac{1}{2}(5.4 + 5.7 + 5 + 4.2) = \frac{1}{2} \cdot 20.3 = 10.15. \end{aligned}$$

For Simpson's rule, we have

$$\begin{aligned} S_4 &= \frac{1}{3}\left(2.4 + 4 \cdot 3 + 2 \cdot 2.7 + 4 \cdot 2.3 + 1.9\right) \\ &= \frac{1}{3}(2.4 + 12 + 5.4 + 9.2 + 1.9) = \frac{1}{3} \cdot 30.9 \approx 10.3. \end{aligned}$$