Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 7
Due Tuesday, March 8

This week's mastery quiz has four topics. Please check Blackboard to see what you still need to submit. This is the first week S6 appears on a quiz, but it did appear on the midterm. This is the last opportunity for M2, S3, and S4.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 4: Improper Integrals
- Secondary Topic 5: Arc Length and Surface Area
- Secondary Topic 6: Differential Equations

Name:

Recitation Section:

M2: Advanced Integration Techniques

(a) $\displaystyle\int (x-1)^3\sqrt{2x-x^2}\,dx =$

Solution:

We're going to set $x-1=\sin\theta$ so that $dx=\cos\theta\,d\theta$. Then

$$\begin{aligned}\int (x-1)^3\sqrt{2x-x^2}\,dx \int (x-1)^3\sqrt{1-(x-1)^2}\,dx\\ &= \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)\,d\theta\\ &= \int \sin^3(\theta)\cos^2(\theta)\,d\theta\\ &= \int \sin(\theta)\cos^2(\theta)-\sin(\theta)\cos^4(\theta)\,d\theta\\ &= -\frac{1}{3}\cos^3(\theta)+\frac{1}{5}\cos^5(\theta)+C.\end{aligned}$$

But

$$\cos(\theta)=\cos(\arcsin(x-1))=\sqrt{1-(x-1)^2}=\sqrt{2x-x^2},$$

so

$$\int (x-1)^3\sqrt{2x-x^2}\,dx=\frac{1}{5}\sqrt{2x-x^2}^5-\frac{1}{3}\sqrt{2x-x^2}^3+C.$$

(b) Compute $\displaystyle\int \sin(2x)e^{5x}\,dx$.

Solution:

$$\begin{aligned}\int \sin(2x)e^{5x}\,dx &= \sin(2x)\frac{e^{5x}}{5}-\int 2\cos(2x)\frac{e^{5x}}{5}\,dx\\ &= \frac{1}{5}\sin(2x)e^{5x}-\frac{2}{5}\left(\cos(2x)\frac{e^{5x}}{5}-\int -2\sin(2x)\frac{e^{5x}}{5}\,dx\right)\\ &= \frac{1}{5}\sin(2x)e^{5x}-\frac{2}{25}\cos(2x)e^{5x}+\frac{4}{25}\int \sin(2x)e^{5x}\,dx\\ \frac{29}{25}\int \sin(2x)e^{5x}\,dx &= \frac{1}{5}\sin(2x)e^{5x}-\frac{2}{25}\cos(2x)e^{5x}+C\\ \int \sin(2x)e^{5x}\,dx &= \frac{5}{29}\sin(2x)e^{5x}-\frac{2}{29}\cos(2x)e^{5x}+C.\end{aligned}$$

S4: Improper Integrals

(a) Compute $\displaystyle\int_1^2 \frac{dx}{x\ln(x)} =$

Solution:

$$\begin{aligned}\int_1^2 \frac{dx}{x\ln(x)} &= \lim_{t\to 1^+}\int_t^2 \frac{dx}{x\ln(x)}\\ &= \lim_{t\to 1^+} \ln(|\ln(x)|)\Big|_t^2\\ &= \lim_{t\to 1^+} \ln(|\ln(2)|) - \ln|\ln(t)|\end{aligned}$$

But $\lim_{t\to 1^+}\ln(t) = 0$, so $\lim_{t\to 1^+}\ln|\ln(t)| = -\infty$. So this limit diverges.

(b) Compute $\int_1^\infty \frac{1}{x^4}\,dx =$

Solution:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^4} &= \lim_{t\to\infty}\int_1^t \frac{dx}{x^4}\\ &= \lim_{t\to\infty} \frac{-1}{3x^3}\Big|_1^t\\ &= \lim_{t\to\infty} \frac{1}{3} - \frac{1}{3t^5} = \frac{1}{3}.\end{aligned}$$

S5: Arc Length and Surface Area

(a) Set up (but don't evaluate!) an integral that gives the arc length of the curve $\arctan(x) = y^3$ as x varies from 1 to 6.

Solution: We have $y = \sqrt[3]{\arctan(x)}$ so $y' = \frac{1}{3}(\arctan(x))^{-2/3}\frac{1}{1+x^2}$, and thus

$$L = \int_1^6 \sqrt{1 + \frac{1}{9\arctan(x)^{4/3}(1+x^2)^2}}\,dx.$$

(b) Compute the area of the surface obtained by taking the curve $y = \sqrt{15-x}$ as x goes from 3 to 5 and rotating it around the x-axis.

Solution: We have $y' = \frac{-1}{2\sqrt{15-x}}$. So we get

$$\begin{aligned}
A &= \int_3^5 2\pi y\sqrt{1+y'^2}\,dx \\
&= \int_3^5 2\pi\sqrt{15-x}\sqrt{1+\frac{1}{4(15-x)}}\,dx \\
&= 2\pi\int_3^5 \sqrt{15-x+\frac{1}{4}}\,dx \\
&= \pi\int_3^5 \sqrt{61-4x}\,dx \\
&= \pi\frac{2}{3\cdot(-4)}(61-4x)^{3/2}\Big|_3^5 = \frac{-\pi}{6}\left(41^{3/2}-49^{3/2}\right) \\
&= \frac{\pi}{6}(343-41\sqrt{41}) =\approx 42.13.
\end{aligned}$$

S6: Differential Equations

(a) Find a general solution to the equation $y' = x^2+1+x^2y+y$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= (x^2+1)(1+y) \\
\frac{dy}{y} &= x^2+1\,dx \\
\ln|1+y| &= {}^{x^3}\!/_3 + x + C \\
1+y &= e^{x^3/3+x+C} \\
y &= e^{x^3/3+x+C}-1 = Ke^{x^3/3+x}-1.
\end{aligned}$$

(b) Find a (specific) solution to the initial value problem $-2x+4y^3\sqrt{x^2+4}\cdot y' = 0$ if $y(0)=2$

Solution:

$$\begin{aligned}
4y^3y'\sqrt{x^2+4} &= 2x \\
4y^3\,dy &= \frac{2x}{\sqrt{x^2+4}}\,dx \\
y^4 &= 2\sqrt{x^2+4}+C \\
y &= \sqrt[4]{2\sqrt{x^2+4}+C}.
\end{aligned}$$

Then we have

$$\begin{aligned} 2 &= \sqrt[4]{2\sqrt{4} + C} = \sqrt[4]{4 + C} \\ 16 &= 4 + C \\ C &= 12 \\ y &= \sqrt[4]{2\sqrt{x^2 + 4} + 12}. \end{aligned}$$