Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 8
Due Tuesday, March 22

This week's mastery quiz has four topics. Please check Blackboard to see what you still need to submit. This is the first week S6 appears on a quiz, but it did appear on the midterm. This is the last opportunity for S6.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 6: Differential Equations
- Secondary Topic 7: Sequences and Series

Name:

Recitation Section:

M3: Series Convergence

Determine whether each of the following series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

Solution: We compute

$$\begin{aligned}\int_1^\infty \frac{x}{x^4+1}\,dx &= \lim_{t\to\infty}\int_1^t \frac{x}{x^4+1}\,dx \\ &= \lim_{t\to\infty}\frac{1}{2}\arctan(x^2)\Big|_1^t = \lim_{t\to\infty}\frac{1}{2}\big(\arctan(t^2)-\arctan(1)\big) \\ &= \frac{1}{2}(\pi/2-\pi/4) = \frac{\pi}{8}.\end{aligned}$$

This is finite and convergent, so by the integral test, the series $\sum_{n=1}^\infty \frac{n}{n^4+1}$ converges.

Alternatively, we could observe that $\frac{n}{n^4+1} \le \frac{1}{n^3}$. Since $\sum_{n=1}^\infty \frac{1}{n^3}$ converges by the p-series test, we can conclude that $\sum_{n=1}^\infty \frac{n}{n^4+1}$ converges by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\ln(n)}$

Solution: By L'Hospital's rule, we compute that

$$\lim_{n\to\infty}\frac{n}{\ln(n)} = \lim_{x\to+\infty}\frac{x^{\nearrow\infty}}{\ln(x)_{\searrow\infty}} \stackrel{\text{L'H}}{=} \lim_{x\to+\infty}\frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Since this isn't zero, the series diverges by the divergence test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$

Solution: We have

$$\begin{aligned}\int_1^\infty \frac{4x^3+1}{x^4+x+3}\,dx &= \lim_{t\to\infty}\int_1^t \frac{4x^3+1}{x^4+x+3}\,dx = \lim_{t\to\infty}\ln(x^4+x+3)\Big|_1^t \\ &= \lim_{t\to\infty}\ln(t^4+t+3)-\ln(5) = \infty.\end{aligned}$$

This diverges, so by the integral test the series $\sum_{i=1}^\infty \frac{4n^3+1}{n^4+n+3}$ diverges.

We can also try to use the Comparison Test here, but it's a little tricky, because $\frac{4n^3+1}{n^4+n+3} < \frac{4}{n}$, and that doesn't help because $\sum_{n=1}^\infty \frac{4}{n} = \infty$. If we want to do comparison, we can try to argue that while $\frac{4n^3+1}{n^4+n+3} < \frac{4}{n}$, it's also true that $\frac{4n^3+1}{n^4+n+3} > \frac{1}{n}$. But that's not super obvious and would take some work.

Or we could use the Limit Comparison Test, and argue that

$$\lim_{n\to\infty}\frac{\frac{4n^3+1}{n^4+n+3}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{4n^4+n}{n^4+n+3} = 4$$

is a finite non-zero limit. Since $\sum_{n=1}^\infty \frac{1}{n}$ diverges (by the p-series test, or because it's the harmonic series), we know by the Limit Comparison Test that $\sum_{n=1}^\infty \frac{4n^3+1}{n^4+n+3}$ diverges.

S6: Differential Equations

(a) Find a general solution to the equation $y' = xe^x y$.

Solution:

$$\begin{aligned}\frac{dy}{dx} &= xye^y \\ \frac{dy}{y} &= xe^x\,dx \ln|y| &&= xe^x - e^x + C \\ y &= e^{xe^x - e^x}e^C.\end{aligned}$$

(b) Find a (specific) solution to the initial value problem $y'/x = \cos^2(y)$ if $y(0) = \pi/3$

Solution:

$$\begin{aligned}\frac{dy}{dx} &= x\cos^2(y) \\ \sec^2(y)\,dy &= x\,dx \\ \tan(y) &= x^2/2 + Cy &&= \arctan(x^2/2 + C).\end{aligned}$$

Then we have

$$\begin{aligned}\pi/3 &= \arctan(C)\tan(\pi/3) &&= CC = \sqrt{3} \\ y &= \arctan(x^2/2 + \sqrt{3}).\end{aligned}$$

S7: Sequences and Series

(a) Let $b_n = \frac{(n)!}{(n+2)!}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$.

Solution:

$b_1 = 1/6, b_2 = 2/24 = 1/12, b_3 = 6/120 = 1/20$, and $b_4 = 24/720 = 1/30$.

We compute

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n^2+3n+2} = 0.$$

(b) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+5n+6}$

Solution:

A partial fractions decomposition tells us that $\frac{2}{n^2+5n+6} = \frac{2}{n+2} - \frac{2}{n+3}$. Then our partial sums are

$$\begin{aligned}\sum_{n=1}^{k} \frac{2}{n^2+5n+6} &= \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) + \cdots + \left(\frac{2}{k+2} - \frac{2}{k+3}\right) = \frac{2}{3} - \frac{2}{k+3} \\ \sum_{n=1}^{\infty} \frac{2}{n^2+5n+6} &= \lim_{k\to\infty} \frac{2}{n^2+5n+6} = \lim_{k\to\infty} \frac{2}{3} - \frac{2}{k+3} = \frac{2}{3}.\end{aligned}$$

(c) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3^{2n}} =$

Solution: This is a geometric series with $a = 4/9$ and $r = 1/9$, so we have

$$\sum_{n=1}^{\infty} \frac{4}{3^{2n}} == \frac{4/9}{1 - 1/9} = 1/2.$$