Math 1232 Spring 2022
Single-Variable Calculus 2 Mastery Quiz 9
Due Tuesday, March 29

This week's mastery quiz has two topics. This week is the last opportunity for S7. It is the second opportunity for M3, so everyone should submit that topic.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Secondary Topic 7: Sequences and Series

Name:

Recitation Section:

M3: Series Convergence

Analyze the convergence of the following three series. (Specify if they converge absolutely, converge conditionally, or diverge.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n + 1}$

Solution:

We use the Ratio test. We have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(-1)^{n+1}3^{n+1}/5^{n+1}+1}{(-1)^n 3^n/5^n+1} \right| &= \lim_{n\to\infty} \frac{3^{n+1}(5^n+1)}{3^n(5^{n+1}+1)} \\ &= \lim_{n\to\infty} 3\frac{5^n+1}{5^{n+1}+1} \\ &= \lim_{n\to\infty} 3\frac{1+1/5^n}{5+1/5^n} = \frac{3}{5}.\end{aligned}$$

This limit is less than 1, so by the ratio test this converges absolutely.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin(n)}{n^3+2}$

Solution: This series has positive and negative terms, but it's not alternating. We basically have to look at absolute convergence.

We consider the series

$$\sum_{n=1}^{\infty} \left| \frac{n\sin(n)}{n^3+2} \right| = \sum_{n=1}^{\infty} \frac{n|\sin(n)|}{n^3+2}.$$

We can't really use the limit comparison test here, because the $\sin(n)$ will screw it up. But we can use the usual comparison test. We know that $0 \le |\sin(n)| \le 1$, so

$$\frac{n|\sin(n)|}{n^3+2} \le \frac{n}{n^3} = \frac{1}{n^2}.$$

We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test, so this series converges by the comparison test. Thus our original series converges absolutely.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Solution:

This is an alternating series. Since the terms $\frac{n}{n^2+1}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You can see this doesn't converge in a couple ways. The integral test would work. The regular comparison test will *not* work unless you're really careful, but $\frac{n}{n^2+1} < \frac{1}{n}$ so we'd need to do some chicanery.

So it seems like this calls for the limit comparison test. We have

$$\lim_{n\to\infty} \frac{n/n^2+1}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Since the harmonic series $\sum \frac{1}{n}$ diverges, by the limit comparison test, $\sum \frac{n}{n^2+1}$ diverges, and thus our series does not converge absolutely.

S7: Sequences and Series

(a) Let $b_n = \frac{n!}{2^n}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

Solution:

$$b_1 = \frac{1}{2} \qquad b_2 = \frac{2}{4}$$
$$b_3 = \frac{6}{8} \qquad b_4 = \frac{24}{16}.$$

We see that

$$\frac{n!}{2^n} = \frac{n(n-1)(n-2)\dots(2)(1)}{2(2)(2)\dots(2)(2)} \geq \frac{n}{2}.$$

Since $\lim_{n\to\infty} \frac{n}{2} = \infty$, we know that $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} =$

Solution:

We can do a partial fractions decomposition: we have

$$\begin{aligned} 2 &= A(n+1) + B(n+3) \\ 2 &= 2B & \Rightarrow B = 1 \\ 2 &= -2A & \Rightarrow A = -1 \end{aligned}$$

so our sum is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \dots \\ &= \frac{1}{2} + \frac{1}{3} = \frac{5}{6}. \end{aligned}$$

More rigorously, we have

$$\begin{aligned}
\sum_{n=1}^{k} \frac{2}{n^2+4n+3} &= \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right) \\
&\quad + \cdots + \left(\frac{1}{k+1}-\frac{1}{k+3}\right) \\
&= \frac{1}{2}+\frac{1}{3}-\frac{1}{k+2}-\frac{1}{k+3} \\
\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} &= \lim_{k\to\infty} \frac{1}{2}+\frac{1}{3}-\frac{1}{k+2}-\frac{1}{k+3} \\
&= \frac{1}{2}+\frac{1}{3} = \frac{5}{6}.
\end{aligned}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^n} =$

Solution:

This is a geometric series with $a = \frac{8}{3}$ and $r = \frac{4}{3}$. Since $r > 1$ this series does not converge, and the sum of the series is ∞.