Math 1231 Practice Midterm Solutions

Instructor: Jay Daigle

Problem 1 (M1)**.** Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(b)

$$\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned}
\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3+\sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6})+x/x^3} \\
&= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\
&= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1.
\end{aligned}$$

(c)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(d)

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 2 (M2)**.** Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)$

Solution:

$$f'(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \tan\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \frac{\frac{1}{2}(x^2+1)^{-1/2}2x(x+2) - \sqrt{x^2+1}}{(x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4} \cdot \frac{(3x^2-\sin(x^2)2x)(\sin(x^3)+1) - \cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

Problem 3 (S1)**.**

Suppose $f(x) = x^2 - 6x$, and we want an output of approximately -9. What input a should we aim for? Find a δ so that if our input is $a \pm \delta$ then our output will be -9 ± 2. Justify your answer.

Solution: We want an input of about $a = 3$. Our output error will be $|x^2 - 6x + 9| = |x-3|^2$. We want this to be less than 2, so we need

$$|x-3|^2 < 2$$
$$|x-3| < \sqrt{2},$$

so we can take $\delta = \sqrt{2}$.

Problem 4 (S2)**. Directly from the definition of derivative,** compute the derivative of $f(x) = x^2 + \sqrt{x}$ at $a = 2$.

Solution:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2 + \sqrt{2+h} - 2^2 - \sqrt{2}}{h} \\
&= \left(\lim_{h\to 0} \frac{4h+h^2}{h}\right) + \left(\lim_{h\to 0} \frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right) \\
&= \left(\lim_{h\to 0} 4+h\right) + \left(\lim_{h\to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}\right) \\
&= 4 + \frac{1}{2\sqrt{2}}.
\end{aligned}$$

Problem 5 (S3)**.** Give equation for the linear approximation of the function $f(x) = x\sin(x)$ near the point $a = \pi/2$. Use it to estimate $f(1.5)$.

Solution: We calculate that $f(\pi/2) = \pi/2 \sin(\pi/2) = \pi/2$, and $f'(x) = \sin(x) + x\cos(x)$, so $f'(\pi/2) = \sin(\pi/2) + \pi/2\cos(\pi/2) = 1$. So

$$f(x) \approx \pi/2 + 1(x - \pi/2) = x.$$

Thus we have

$$f(1.5) \approx 1.5.$$

(The true answer is $1.49624\ldots$.)