Math 1231 Practice Midterm Solutions

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Problem 1 (M3). (a) Find and classify all the critical points of $f(x) = (x-5)\sqrt[3]{x^2}$. [Note: this is quite hard but it's good practice.]

Solution: We compute

$$\begin{aligned} f'(x) &= \sqrt[3]{x^2} + (x-5)\frac{2}{3}x^{-1/3} = x^{2/3} + \frac{2x-10}{3\sqrt[3]{x}} \\ &= \frac{3x+2x-10}{3\sqrt[3]{x}} = 5\frac{x-2}{3\sqrt[3]{x}} \end{aligned}$$

This is equal to zero when $x = 2$ and is undefined when $x = 0$, so the two critical points are $x = 0$ and $x = 2$.

We could try to use the second derivative test here, but it won't really work. We get

$$\begin{aligned} f''(x) &= \frac{10x+10}{9x^{4/3}} \\ f''(2) &= \frac{5}{3\sqrt[3]{2}} > 0 \end{aligned}$$

so we see that f has a local minimum at $x = 2$, but $f''(0)$ is undefined so it doesn't help us classify $x = 0$.

Instead we compute a chart

	$5(x-2)$	$\frac{1}{3\sqrt[3]{x}}$	$f'(x)$
$x < 0$	$-$	$-$	$+$
$0 < x < 2$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$

Thus we conclude that f has a local maximum at $x = 0$ and a local minimum at $x = 2$.

(b) The function $g(x) = (x^2 - 3x)\sqrt[3]{x-3}$ has absolute extrema either on $(-4, -1)$ or on $[1, 4]$. Pick one of those intervals, explain why g has extrema on that interval, and find the absolute extrema.

Solution: We compute

$$g'(x) = (2x-3)\sqrt[3]{x-3} + (x^2-3x)\frac{1}{3}(x-3)^{-2/3}.$$

This is undefined when $x = 3$, and to find the zeroes we compute

$$\begin{aligned} 0 &= (2x-3)\sqrt[3]{x-3} + (x^2-3x)\frac{1}{3}(x-3)^{-2/3} \\ &= (2x-3)\sqrt[3]{x-3} + \frac{x^2-3x}{3(x-3)^{2/3}} \\ 0 &= (2x-3)\sqrt[3]{x-3} \cdot 3(x-3)^{2/3} + (x^2-3x) \\ &= (2x-3)3(x-3) + (x^2-3x) = 3(2x^2-3x-6x+9) + x^2 - 3x \\ &= 7x^2 - 30x + 27 = (7x-9)(x-3) \end{aligned}$$

which is zero when $x = 3$ or $x = 9/7$. So then we have

$$\begin{aligned} g(1) &= -2\sqrt[3]{-2} = 2\sqrt[3]{2} \\ g(9/7) &= \left(\frac{91}{49} - \frac{27}{7}\right)\sqrt[3]{-12/7} = \frac{108}{49}\sqrt[3]{12/7} \\ g(3) &= 0 \\ g(4) &= 4 \cdot \sqrt[3]{1} = 4. \end{aligned}$$

All of these numbers are non-negative, so the minimum value is 0, at $x = 3$. We can see that $2\sqrt[3]{2} < 2 \cdot 2 = 4$, and maybe convince ourselves that

$$\frac{108}{49}\sqrt[3]{12/7} \approx 2 \cdot 1 < 4.$$

Thus the maximum value is 4, which occurs at 4.

Problem 2 (S4)**.** Suppose that $Q(p) = 3p^2 + 10p - 100$ is the number of widgets you can buy at a price of p dollars.

(a) What are the units of $Q'(p)$? What does it represent physically? What does it mean if $Q'(p)$ is big?

(b) Calculate $Q'(10)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution:

(a) $Q'(p)$ has units of widgets per dollar. The derivative is the rate at which increasing the price increases the number of widgets you can buy (called the marginal elasticity of demand, though you don't need to know that on the test). If $Q'(p)$ is big, that means that raising your price by 1 dollar gets you a lot more widgets available to buy.

(b) $Q'(p) = 6p + 10$ so $Q'(10) = 70$. This means that if you are buying widgets for \$10, you can get approximately seventy more widgets if you raise your price to \$11.

Problem 3 (S5)**.** Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$\begin{aligned} 4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} &= 0 \\ 4x^3 - 4xy^2 &= 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx} \\ \frac{4x^3 - 4xy^2}{4x^2y - 4y^3} &= \frac{dy}{dx} \end{aligned}$$

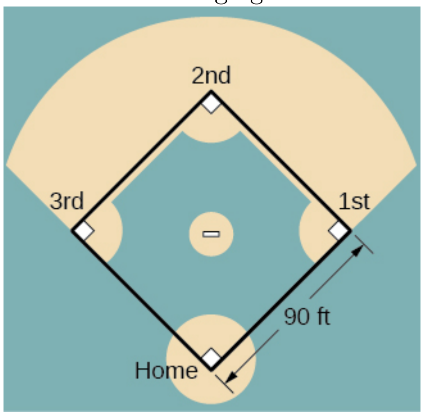
Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5} \cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4 \cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 1 &= \sqrt{5}(x - \sqrt{5}). \end{aligned}$$

Problem 4 (S6)**.** Consider this baseball diamond, which is a square with 90ft sides. A batter hits the ball and runs from home toward first base at a speed of 22ft/s. At what rate is the distance between the runner and second base changing when the runner has run 30ft?



Solution: We use the Pythagorean theorem, $a^2 + b^2 = c^2$, where a is the distance of the runner from first base and b is the distance of second base from first base. Then c is the distance between the runner and second base, which we want to know about, and we have it related to a and b, which we do know about.

When the runner has run 30ft, then we have $a = 60$ft and $b = 90$ft is a constant. Then we have

$$\begin{aligned} c^2 &= a^2 + b^2 = 60^2 + 90^2 = 3600 + 8100 = 11700 \\ c &= 10\sqrt{117} = 30\sqrt{13}. \end{aligned}$$

Alternatively,

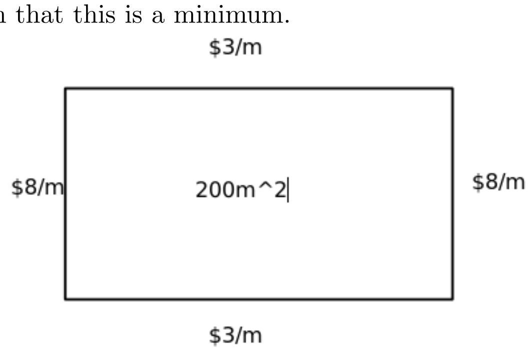
$$\begin{aligned} c^2 &= 60^2 + 90^2 = 30^2(2^2 + 3^2) \\ c &= 30\sqrt{13}. \end{aligned}$$

We know that $a' = -22$ft/s, so we compute

$$\begin{aligned} 2aa' + 2bb' &= 2cc' \\ aa' + bb' &= cc' \\ 60\text{ft} \cdot (-22)\text{ft/s} + 90\text{ft} \cdot 0\text{ft/s} &= 30\sqrt{13}\text{ft} \cdot c' \\ 2 \cdot -22\sqrt{13}\text{ft/s} &= c'. \end{aligned}$$

So the distance between the runner and second base is decreasing at $\frac{44}{\sqrt{13}} \approx 12.2$ feet per second.

Problem 5 (S8)**.** We want to build a rectangular fence that will enclose 200m^2. One pair of parallel sides cost \$3/m and the other pair costs \$8/m. What dimensions minimize the cost of the fence? Justify your claim that this is a minimum.



Solution: We want to minimize $3w + 8\ell$ subject to the constraint $\ell w = 200$. Thus we have $w = 200/\ell$, and then our function is $C(\ell) = 600/\ell + 8\ell$. We get

$$\begin{aligned} C' &= -600/\ell^2 + 8 = 0 \\ -8\ell^2 &= -600 \\ \ell^2 &= 75 \\ \ell &= 5\sqrt{3}. \end{aligned}$$

Then we have $\ell = 5\sqrt{3}$ and $w = \frac{200}{5\sqrt{3}} = \frac{40}{3}\sqrt{3}$.

We have two options for proving this is a maximum (we only need one):

(a) Extreme Value Theorem: We can't really use the EVT here because we don't have a closed interval.

(b) First Derivative Test: For $\ell < 5\sqrt{3}$ we have $C' < 0$ so the function is decreasing, and for $\ell > 5\sqrt{3}$ we have $C' <>$ so the function is increasing. Thus we have a unique minimum at $5\sqrt{3}$.

(c) Second derivative test: $C''(\ell) = -1200/\ell^3$. Then $C''(\ell) > 0$ for $\ell?0$, which implies there is a single relative minimum, at $\ell = 5\sqrt{3}$. This doesn't really rigorously prove that this is an absolute maximum but I'll take it.

Problem 6 (S7)**.** Let $f(x) = \dfrac{x^3-2}{x^4}$. We compute that $f'(x) = \dfrac{8-x^3}{x^5}$ and $f''(x) = \dfrac{2x^3-40}{x^6}$. Sketch a graph of f.

Your answer should discuss the domain, asymptotes, roots, limits at infinity, critical points and values, intervals of increase and decrease, and concavity.

Solution: The function is defined everywhere except at 0. Near zero, we can see the top is always negative and the bottom is always positive, so $\lim_{x\to 0} f(x) = -\infty$ and we should have a downwards asymptote on either side.

We see there is a root at $x = \sqrt[3]{2}$, and $\lim_{x\to\pm\infty} f(x) = 0$.

We see that $f'(x)$ is undefined at $x = 0$, and is zero when $x^3 = 8$ and thus when $x = 2$. So our critical points occur at 0 and 2. We calculate $f(2) = \frac{6}{16}$, and f isn't defined at 0. By making a chart, we get

	$8-x^3$	x^5	$f'(x)$
$x < 0$	$+$	$-$	$-$
$0 < x < 2$	$+$	$+$	$+$
$2 < x$	$-$	$+$	$-$

so f is decreasing for values less than zero or greater than 2, and increasing for values between 0 and 2.

The second derivative is undefined at 0, and is zero when $2x^3 - 40$ and so when $x = \sqrt[3]{20}$, so our potential points of inflection are 0, $\sqrt[3]{20}$. We compute $f(\sqrt[3]{20}) = \frac{18}{20\sqrt[3]{20}}$. We can make a chart again, but we see that the denominator of $f''(x) \geq 0$, so $f''(x) > 0$ if $x > \sqrt[3]{20}$ and $f''(x) < 0$ if $x < \sqrt[3]{20}$.