Math 1231: Single-Variable Calculus 1
George Washington University   Fall 2023
Recitation 1

Jay Daigle

August 30, 2023

Review Questions

These are all questions on material you should have learned in pre-calculus, but that you might be rusty on. Give them a try and talk to each other! But don't use a calculator; that will get in the way of what you should be learning.

(a) What are the solutions to $x^2 + 5x + 6 = 0$?

Solution: We have $x^2 + 5x + 6 = (x + 2)(x + 3)$, so the solutions are $x = -2$ and $x = -3$.

(b) What are the solutions to $x^2 + 5x + 5 = 0$?

Solution: We can't really factor this, so instead we use the quadratic formula. So the solutions are

$$x = \frac{-5 \pm \sqrt{25 - 20}}{2} = \frac{-5}{2} \pm \frac{\sqrt{52}}{.}$$

(c) Compute $(3x + 5)^2$.

Solution: You can FOIL this, or know the formula for squaring a polynomial. either way the answer is $9x^2 + 30x + 25$.

Importantly the answer is *not* ~~$9x^2 + 25$~~.

(d) Factor $x^3 - 8$.

Solution: This relies on a "difference of cubes" formula. There is a rule that $a^3 - b^3 = (a-b)(a^2+ab+b^2)$, so $x^3 - 8 = (x-2)(x^2+2x+4)$.

(e) Compute $\sin(\pi/3)$.

Solution: $\sin(\pi/3) = \sqrt{3}/2$.

(f) Compute $\tan(5\pi/6)$.

Solution: $\sin(5\pi/6) = 1/2$ and $\cos(5\pi/6) = -\sqrt{3}/2$ so $\tan(5\pi/6) = \frac{1/2}{-\sqrt{3}/2} = \frac{1}{-\sqrt{3}}$. You may want to clean that answer up to $\frac{-1}{\sqrt{3}}$ or even $\frac{\sqrt{3}}{3}$ but the first version is fine.

(g) Compute $\sec(-\pi/4)$.

Solution: $\cos(-\pi/4) = \sqrt{2}/2$ so $\sec(-\pi/4) = \frac{2}{\sqrt{2}}$. You could also clean this up and just say $\sec(-\pi/4) = \sqrt{2}$.

Estimation

The first thing we're going to discuss in class is the idea of *estimation*. Most of the time we compute things exactly: $3 \times 5 = 15$. Other times we estimate: $32 \times 49 \approx 1500$, but this isn't exactly true.

But I want to raise a new question: suppose we want to get our output really close to some target. So we want, e.g., to get 1500 ± 25. We can see that 32×49 is close-ish to 1500, but not that close!

Problem 1. A cylindrical water tank has a base with an area of six square inches. We want to fill it with water ten inches deep, to the nearest inch.

(a) What is the exact volume of water would we ideally want to pour in?

(b) What is the least height that would satisfy our requirements? What is the least volume of water we can pour into the tank and still satisfy our height requirements?

(c) What is the greatest height that would satisfy our requirements? What is the greatest volume of water we can pour into the tank and satisfy our height requirements?

(d) How much error can we have in the volume? That is, how far away from our ideal volume in part (a) can we get and still be safe?

Solution:

(a) 60 cubic inches.

(b) The least height that would satisfy our requirements is 9.5 inches. (This is a bit of a trick question; if we're rounding we need 9.5 or more, and 9 won't work.) The volume that would give us this height is 57 cubic inches.

(c) The greatest height that would work is 10.5 inches, and the volume that would give this height is 63 cubic inches.

(d) The error in our volume can be at most 3 cubic inches: $63 - 60 = 3$ and also $60 - 57 = 3$.

Problem 2. We know that $\sqrt{4} = 2$. Suppose we want to have $1 \leq \sqrt{x} \leq 3$. We might describe this as wanting $\sqrt{x} = 2 \pm 1$. If we're being fancy, we use the Greek letter ε and say that we want an error tolerance of $\varepsilon = 1$.

(a) What is the largest input that keeps the output within ε of 2?

(b) What is the smallest input that keeps the output within ε of 2?

(c) Our ideal input is 4. How much can we miss our ideal input by? (When we're being fancy, we call this number by the Greek letter δ).

Now instead let's take $\varepsilon = .5$.

(d) What is the largest input that keeps the output within ε of 2?

(e) What is the smallest input that keeps the output within ε of 2?

(f) What does that make δ?

Solution:

(a) 9

(b) 1

(c) This is actually kind of a trick question. You *could* say that the inputs need to be 5 ± 4 so $\delta = 4$, but that's not how we want to think about it in this course. (Anyone who came up with that answer isn't really wrong; they're just answering a slightly different question.)

Since the input that gives us the *exact* answer we want is 4, we're looking for $4 \pm \delta$. We can overshoot our ideal by 5, but we can only undershoot it by 3. So if we don't know which direction we're going to miss in, we're only safe if we miss by 3 or less. So we set $\delta = 3$.

(You could imagine, in a real-world process, choosing to aim for 5. You'd accept being too high on average in exchange for it being easier to stay within the error margin overall. But that's not how we want to set this up.)

(d) 6.25

(e) 2.25

(f) On the low end, we can miss by 1.75; on the high end we can miss by 2.25. So we take $\delta = 1.75$. (Again, you "could" take 4.25 ± 2, but that's answering a slightly different question.)

In class, we looked at the following question: Suppose we want to make a square platform that's 16 square meters, plus or minus 1. How long do the sides need to be?

Clearly, our sides need to be between $\sqrt{15}$ and $\sqrt{17}$ but that doesn't tell us anything useful. So instead we made the following argument: We can use an absolute value to describe the way we think about errors. in particular, what we want here is

$$|s^2 - 16| < \varepsilon = 1, \tag{1}$$

and factoring the left hand side gives $|s-4| \cdot |s+4| < 1$. We can't solve this exactly, but we can make the following lazy decision: We know s should be *approximately* 4. It might be a little bigger, so $s + 4$ might be bigger than 8, but it's certainly less than 9, or 10. Then we just need to solve

$$|s-4| \cdot |s+4| < 10|s-4| < 1 \tag{2}$$

$$|s-4| < .1 \tag{3}$$

$$-.1 < s - 4 < .1 \tag{4}$$

$$3.9 < s - 4 < 4.1. \tag{5}$$

Thus $\delta = .1$ and s should be $4 \pm .1$.

This is a tricky argument! But I want you to try to think through it now.

Problem 3. Let's suppose instead we want to make a square platform with area 25 square meters, plus or minus 1.

1. Write down the analogue of inequality (1) for this new problem. Can you explain in words what this inequality says about your error?
2. We can factor the left-hand side of this inequality into two factors. If our input is close to 5, one of these terms will be small, and the other will be large. Which one will be large, and about how large will that be?
3. This should let you write down an inequality like the one in (2). What is it?
4. Figure out δ such that $s = 5 \pm \delta$ will keep us in our error bounds.
5. Check your answer: square $5 + \delta$ and $5 - \delta$ and see whether the answers fall within your error margin.
6. Could you use a larger δ than the one you found in part (4)?

Solution:

1. $|s^2 - 25| < 1$. This says that the error between our output s^2 and our target 25 is less than one.
2. We get $|s - 5| \cdot |s + 5| < 1$. The $|s - 5|$ term should be small since we want s close to 5; the $|s + 5|$ term will be large, and it should be approximately 10 since $s \approx 5$.
3. $s + 5 \approx 10$ so we can say $s + 5 < 11$. So we get $|s - 5| \cdot |s + 5| < 11|s - 5| < 1$.
4. Then we need $|s - 5| < 1/11$ which gives us $\delta = 1/11$.
5. $(54/11)^2 \approx 24.0992$ and $(56/11)^2 \approx 25.9174$ so $\delta = 1/11$ is in fact an acceptable amount of error in the input.
6. We see that $4.9^2 = 24.01$ keeps us within our error margin; but $5.1^2 = 26.01$ does not. So $\delta = 1/10$ is too big. However, we could take something like $\delta = .095$, which is bigger than $1/11 \approx .091$. Then $4.905^2 = 24.059$ and $5.095^2 = 25.959$ both stay within our error margin.

   The largest *possible* δ that works is $\sqrt{26} - 5 \approx .099$. But it's hard to figure that out without already knowing the value of $\sqrt{26}$.