

Math 1231: Single-Variable Calculus 1
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Recitation 3

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Problem 1. We want to compute $\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3}$.

- (a) What rule do we know we need to invoke here?
- (b) What θ are we going to need to pick for this to work out, and why?
- (c) Do algebra so that you can invoke the small angle approximation. What is the limit? (Are you using the AIF property?)
- (d) Go back to the beginning, and see what our heuristic idea that $\sin(\theta) \approx \theta$ would have told you. Does that match with what you got?

Solution:

- (a) We need to use the small angle approximation, because this problem requires us to make trig interact with algebra.
- (b) We basically have to take $\theta = x^2 - 9$, because that's what's inside the sin.
- (c) We need to get a θ on the bottom as well. So we take

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &=_{AIF} \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} \cdot \frac{x + 3}{x + 3} = \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)(x + 3)}{x^2 - 9} \\ &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x^2 - 9} \cdot \lim_{x \rightarrow 3} x + 3 = 1 \cdot (3 + 3) = 6. \end{aligned}$$

You have to use AIF in that first equality, because you're making the function undefined at -3 .

There are a couple different ways to make this algebra work out. You might also try

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &=_{AIF} \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)/x^2 - 9 \cdot x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} =_{AIF} \lim_{x \rightarrow 3} x + 3 = 6,\end{aligned}$$

which is essentially the same logic.

(d)

We have a $\sin(0)$ on the top and a 0 on the bottom, but the 0s don't come from the same form; we need to get a $x^2 - 9$ term on the bottom. Multiplication by the conjugate gives

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x - 3} \cdot \frac{x + 3}{x + 3} = \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)(x + 3)}{x^2 - 9} \\ &= \lim_{x \rightarrow 3} \frac{\sin(x^2 - 9)}{x^2 - 9} \cdot \lim_{x \rightarrow 3} x + 3 = 1 \cdot (3 + 3) = 6.\end{aligned}$$

Problem 2. We want to think about the ways that infinity doesn't really work like a number, and we can't do arithmetic with it.

- To start: what is $\lim_{x \rightarrow 0} 1/x$, and why?
- Let's look at $\lim_{x \rightarrow 0} 1/x + 1/x$. If we computed the limit of each fraction individually, what indeterminate form would we get?
- How do we actually compute $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x}$? (Hint: combine them into one fraction.) Does this make sense in light of what you got in part (b)?
- Now consider $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2}$. What is the limit of each piece, and what indeterminate form is this?
- Compute $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x-1}{x-x^2}$ directly. Does this make sense in light of what you got in part (d)?
- Now consider $\lim_{x \rightarrow 0} 1/x + 1/x^2$. What indeterminate form would this represent? What is the limit? Do those make sense together?
- Finally, let's look at $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x^2-3x+2}{x^2-2x}$. What indeterminate form is this? What is the limit?
- What pattern do you see from all of these?

Solution:

(a) $\lim_{x \rightarrow 0} \frac{1}{x} = \pm\infty$.

(b) This looks like $\infty + \infty$ as an indeterminate form.

(c) We see $\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x} = \lim_{x \rightarrow 0} \frac{2}{x} = \pm\infty$. This seems to make sense; $\infty + \infty = \infty$ is perfectly reasonable.

(d) We already know $\lim_{x \rightarrow 0} \frac{1}{x} = \pm\infty$. We can compute that

$$\lim_{x \rightarrow 0} \frac{x - 1}{x - x^2} = \pm\infty.$$

So this is again $\infty + \infty$.

(e) By combining fractions, we get

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x - 1}{x - x^2} = \lim_{x \rightarrow 0} \frac{1 - x}{x - x^2} + \frac{x - 1}{x - x^2} = \lim_{x \rightarrow 0} 0 = 0.$$

So here $\infty + \infty = 0$.

(f) We have a $\pm\infty$ plus a $+\infty$, so we get $\infty + \infty$ again. When we combine them into one term we get

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{1}{x^2} = \lim_{x \rightarrow 0} \frac{x + 1}{x^2} = +\infty$$

since the denominator is $x^2 \geq 0$. So here $\infty + \infty = +\infty$.

We could heuristically say that $\frac{1}{x^2}$ goes to $+\infty$ “faster” than $\frac{1}{x}$ goes to $\pm\infty$, and so it wins out; but this is really vague and handwavy so we try to replace it with more precise arguments like this one.

(g) We compute $\lim_{x \rightarrow 0} \frac{x - 3x + 2}{x^2 - 2x} = \pm\infty$, so this is, again, $\infty + \infty$. The actual limit is

$$\lim_{x \rightarrow 0} \frac{1}{x} + \frac{x^2 - 3x + 2}{x^2 - 2x} = \lim_{x \rightarrow 0} \frac{x - 2 + x^2 - 3x + 2}{x^2 - 2x} = \lim_{x \rightarrow 0} \frac{x^2 - 2x}{x^2 - 2x} = \lim_{x \rightarrow 0} 1 = 1.$$

So here $\infty + \infty = 1$.

(h) In conclusion, if you know something looks like $\infty + \infty$, you don’t really know anything about it at all.

Problem 3. (a) Consider $\lim_{x \rightarrow -\infty} \frac{x}{x+1}$. Can you come up with a heuristic guess about what this limit is?

- (b) Can you carefully justify your guess from part (a).
- (c) Now consider $\lim_{x \rightarrow +\infty} \frac{x}{\sqrt{3x^2+1}}$, and come up with a heuristic estimate for the limit.
- (d) Carefully justify your guess from part (c).
- (e) How would either of those calculations change if we take the limit to the other infinity?

Solution:

- (a) If x is large, x and $x + 1$ should behave basically the same; the 1 is insignificant compared to the x . So this limit should be 1.

(b)

$$\lim_{x \rightarrow -\infty} \frac{x}{x+1} = \lim_{x \rightarrow -\infty} \frac{1}{1 + \frac{1}{x}} = \lim_{x \rightarrow -\infty} \frac{1}{1} = 1.$$

- (c) When x is large, the x will be really large, but really small relative to the $3x^2$. So this should look like $\frac{x}{\sqrt{3x^2}}$ which goes to $\frac{1}{\sqrt{3}}$.

(d)

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{3x^2+1}/x} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{3x^2+1}/\sqrt{x^2}} \\ &= \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{3 + \frac{1}{x^2}}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

- (e) The first wouldn't change at all. The second would change, because if $x > 0$ then $x = \sqrt{x^2}$, but if $x < 0$ then $x = -\sqrt{x^2}$. So we instead get

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{x}{\sqrt{3x^2+1}} &= \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{3x^2+1}/x} \\ &= \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{3x^2+1}/-\sqrt{x^2}} \\ &= \lim_{x \rightarrow -\infty} \frac{-1}{\sqrt{3 + \frac{1}{x^2}}} = \frac{-1}{\sqrt{3}}. \end{aligned}$$

Problem 4.

- (a) We want to compute $\lim_{x \rightarrow +\infty} \sqrt{x^2+1} - x$. Can we just plug in here, or is this an indeterminate form? Why?

- (b) When we have an indeterminate form, we generally want to write it as a big fraction, simplify, and factor. How can we do that here? We have to use a technique from last week to really get this to work.
- (c) Once you have a big fraction, use it to compute the limit.
- (d) How does this argument change if instead we want $\lim_{x \rightarrow +\infty} \sqrt{x^2 + x + 1} - x$?
- (e) What is $\lim_{x \rightarrow +\infty} \sqrt{x^2 + ax + 1} - x$?
- (f) What does the answer in part (e) say about $\lim_{x \rightarrow +\infty} \sqrt{x^2 + 2x + 1} - x$? Why should the answer to this question be obvious?

Solution:

- (a) This is indeterminate, of the form $\infty - \infty$.
- (b) The simplest thing we could do is just write

$$\lim_{x \rightarrow +\infty} \sqrt{x^2 + 1} - x = \lim_{x \rightarrow +\infty} \frac{\sqrt{x^2 + 1} - x}{1}.$$

But that doesn't get us very far. Since we have a difference of square roots, we want to multiply by the conjugate.

- (c) We get

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sqrt{x^2 + 1} - x &= \lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + 1} - x \right) \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{(x^2 + 1) - x^2}{\sqrt{x^2 + 1} + x} = \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{x^2 + 1} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{1/x}{\sqrt{1 + 1/x^2} + 1} = 0. \end{aligned}$$

- (d)

$$\begin{aligned} \lim_{x \rightarrow +\infty} \sqrt{x^2 + x + 1} - x &= \lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + x + 1} - x \right) \frac{\sqrt{x^2 + x + 1} + x}{\sqrt{x^2 + x + 1} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{x^2 + x + 1 - x^2}{\sqrt{x^2 + x + 1} + x} = \lim_{x \rightarrow +\infty} \frac{x + 1}{\sqrt{x^2 + x + 1} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{1 + 1/x}{\sqrt{1 + 1/x + 1/x^2} + 1} = \frac{1}{2}. \end{aligned}$$

(e) We can make essentially the same argument:

$$\begin{aligned}\lim_{x \rightarrow +\infty} \sqrt{x^2 + ax + 1} - x &= \lim_{x \rightarrow +\infty} \left(\sqrt{x^2 + ax + 1} - x \right) \frac{\sqrt{x^2 + ax + 1} + x}{\sqrt{x^2 + ax + 1} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{x^2 + ax + 1 - x^2}{\sqrt{x^2 + ax + 1} + x} = \lim_{x \rightarrow +\infty} \frac{ax + 1}{\sqrt{x^2 + ax + 1} + x} \\ &= \lim_{x \rightarrow +\infty} \frac{a + 1/x}{\sqrt{1 + a/x + 1/x^2} + 1} = \frac{a}{2}.\end{aligned}$$

(f) By the answer from part (b), $\lim_{x \rightarrow +\infty} \sqrt{x^2 + 2x + 1} - x = 2/2 = 1$. But we could also just observe that $x^2 + 2x + 1 = (x + 1)^2$, so

$$\lim_{x \rightarrow +\infty} \sqrt{x^2 + 2x + 1} - x = \lim_{x \rightarrow +\infty} (x + 1) - x = 1.$$