Linear equations and matrix row reduction

LC: $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$

$$a_1\begin{bmatrix}1\\2\\3\end{bmatrix} + a_2\begin{bmatrix}1\\2\\4\end{bmatrix} + a_3\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\5\\7\end{bmatrix}$$

$$a_1 + a_2 + a_3 = 2$$
$$2a_1 + 2a_2 + a_3 = 5$$
$$3a_1 + 4a_2 + a_3 = 7$$

$$\left[\begin{array}{cc|c} 1 & 3 & 3 \\ -2 & -5 & -2 \\ -5 & -4 & 7 \\ -3 & -9 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 11 & 22 \\ 0 & 0 & 17 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 3 & 3 \\ -2 & -5 & -2 \\ -5 & -4 & 7 \\ -3 & -9 & 8 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 11 & 22 \\ 0 & 0 & 17 \end{array}\right]$$

$$\begin{array}{ccc} -1 & -3 & -3 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 0 \end{array}$$

$$\to \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -22 \\ 0 & 0 & 17 \end{array}\right] \xrightarrow{II, III} \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

Row echelon form

- Every non-zero row is above every zero row
- The first non-zero of each row is to the right of the first non-zero entry of the above row.
- (1st non-zero entry of each row is 1)

$$\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 3 & -1 & 4 \\ 0 & 0 & -2 & 3 \end{bmatrix} \quad \begin{bmatrix} 5 & 1 & 3 & 2 & 8 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -7 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 5 \\ 0 & -2 & 3 \\ 0 & 0 & 7 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & 5 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 5 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Reduced Row Echelon Form

- REF

  and

- 1st non-zero entry in each row is the only non-zero entry in its column

$$\left[\begin{array}{ccc|c} 4 & 2 & 2 & 8 \\ 3 & 2 & 1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 6 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x+y+z&=1 \\ y+z&=1 \\ z&=1 \\ 0&=0 \end{aligned} \qquad \begin{aligned} x&=0 \\ y&=0 \\ z&=1 \\ 0&=0 \end{aligned}$$

§2.6 Spanning and Linear Independence

$$\underline{a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

Dfn: V a vs/$\mathbb{F}$, $S\subseteq V$. The <u>span</u> of S is the set

$$\{a_1\vec{v}_1+\cdots+a_n\vec{v}_n \mid a_i\in\mathbb{F}, \vec{v}_i\in S\}$$

of LCs of vectors in S.

span(S).

ex: $V=\mathbb{R}^3$, $S=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

$$\text{Span}(S)=\left\{a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

$$=\left\{\begin{bmatrix}a\\b\\0\end{bmatrix}\right\}=\mathbb{R}^2\subseteq\mathbb{R}^3$$

$T=\left\{\begin{bmatrix}3\\2\\0\end{bmatrix},\begin{bmatrix}13\\7\\0\end{bmatrix}\right\}$

$$\text{Span}(T)=\left\{a\begin{bmatrix}3\\2\\0\end{bmatrix}+b\begin{bmatrix}13\\7\\0\end{bmatrix}\right\}=\left\{\begin{bmatrix}3a+13b\\2a+7b\\0\end{bmatrix}\right\}$$

$$T = \left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 13 \\ 7 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^2$$

$$\operatorname{Span}(S) \subseteq \operatorname{Span}(T)$$

$$\operatorname{Span}(T) = \left\{ a \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 13 \\ 7 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 3a + 13b \\ 2a + 7b \\ 0 \end{bmatrix} \right\}$$

$$a \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 13 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 13 & 3 \\ 2 & 7 & 18 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & 13 & x \\ 2 & 7 & y \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 6 & x-y \\ 2 & 7 & y \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 6 & x-y \\ 0 & -5 & 3y-2x \\ 0 & 0 & 0 \end{array}\right]$$

always has a soln

$$x\begin{bmatrix}1\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3a+13b\\2a+7b\\0\end{bmatrix}$$

$$a\begin{bmatrix}3\\2\\0\end{bmatrix}+b\begin{bmatrix}13\\7\\0\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\left[\begin{array}{cc|c}3 & 13 & x\\2 & 7 & y\\0 & 0 & z\end{array}\right]$$

Prop: $V/\mathbb{F}$, $S \subseteq V$.

Then $\text{Span}(S)$ is a subspace of V.

Pf| If $S = \emptyset$, then $\text{span}(S) = \{\vec{0}\}$ is a SS.

Now assume S is non-empty. $S \subseteq V$, so if $\vec{v}_1, \dots, \vec{v}_n \in S$,

$a_1\vec{v}_1 + \dots + a_n\vec{v}_n \in V$. So $\text{Span}(S) \subseteq V$.

1) Let $\vec{v} \in S$. then $0 \cdot \vec{v} \in \text{span}(S)$, and $0 \cdot \vec{v} = \vec{0}$. So $\vec{0} \in \text{span}\, S$.

2) Suppose $\vec{v}_1, \vec{v}_2 \in \text{span}(S)$.

then $\vec{v}_1 = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$ for some $\vec{u}_i \in S$, $a_i \in \mathbb{F}$.

$\vec{v}_2 = b_1\vec{w}_1 + \cdots + b_m\vec{w}_m$ for some $\vec{w}_i \in S$, $b_i \in \mathbb{F}$

So $\vec{v}_1 + \vec{v}_2 = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n + b_1\vec{w}_1 + \cdots + b_m\vec{w}_m$ is a LC of elts of S

so $\vec{v}_1 + \vec{v}_2 \in \text{span}(S)$.

3) Suppose $\vec{v} \in \text{span}\, S$, $r \in \mathbb{F}$.

then $\vec{v} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$ for $\vec{u}_i \in S$, $a_i \in \mathbb{F}$

so $r\vec{v} = r(a_1\vec{u}_1 + \cdots + a_n\vec{u}_n) = (ra_1)\vec{u}_1 + \cdots + (ra_n)\vec{u}_n \in \text{span}(S)$

Cor: $V/\mathbb{F}$, W a SS of V, $S \subseteq V$.

if W contains S, then it contains $\operatorname{span}(S)$.

Pf/ $S \subseteq W$, so $\operatorname{span}(S)$ is a SS of W.

Cor: $V/\mathbb{F}$, $S \subseteq V$, then span(S)
is the smallest SS of V containing S.

Pf/ S is a SS of V: ✓
if $\vec{v} \in S$, $\vec{v} = 1 \cdot \vec{v} \in$ span(S) ✓
If W SS of V, and $S \subseteq W$, then span$(S) \subseteq W$.
So W as big as span(S).

Rmk: span$(S) = \cap$ of all SS containing S.

Dfn: $V/\mathbb{F}$, $S \subseteq V$.

If $\operatorname{span}(S) = V$, then we say

S <u>spans</u> V, or <u>generates</u> V

S is a <u>spanning set</u> for V.

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ex: $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\operatorname{span}(S) = \mathbb{R}^3$$

Dfn: $V/\mathbb{F}$, $S \subseteq V$.

S is ***linearly independent*** if,

for any finite collection $\vec{v}_1, \dots, \vec{v}_n \in S$,

the only scalars solving

$$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0}$$

are $a_1 = \cdots = a_n = 0$.

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idea here: a LI set

has no redundancy.

If S is not LI,

it's ***linearly dependent***

and the eqn

$$a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{0}$$ is a

linear dependence.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Claim S is LI:

suppose $a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

then $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $a_1 = 0$, $a_2 = 0$, $a_3 = 0$.

$$T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

and not all coeffs are 0

so this is LD.