Math 1231
Section 1: Limits

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Subsection 2: Approximation

Approximation Example 1

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- Want: current of 5 amps, to within half of an amp.
- Have: resistance of 2 ohms.

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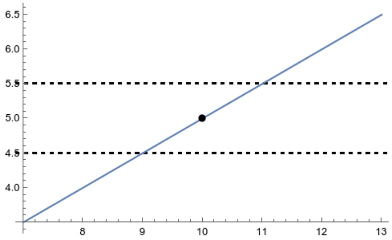
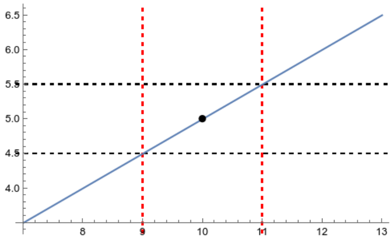
Or V in $(9, 11)$
Or $V = 10 \pm 1$.

Approximation Example 1, Part 2





Left: We want our output to stay between the black dashed lines. Right: If our input stays between the red dashed lines, we'll hit that error threshold.

Approximation Example 1, Part 3

Focus on the *error*.

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Try 1

$$l - 5 < .5$$

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Problem: What if l is small?

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Problem: What if l is small?

Try 2

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If we want $l = 5 \pm .1$, we want $\varepsilon = .1$.

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Need $V = 10 \pm .2$

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Need $V = 10 \pm .2$ so $\delta = .2$.

Approximation Example 2

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- Want: square platform, 16m^2, to within 1m^2.

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So...what is δ?

$$\delta = \min(4 - \sqrt{15}, \sqrt{17} - 4)$$

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$$\begin{aligned} \delta &= \min(4 - \sqrt{15}, \sqrt{17} - 4) \\ &\approx \min(0.127017, 0.123106) \\ &= 0.127017. \end{aligned}$$

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Awkward!

Approximation Example 2 Part 2

Error and epsilon

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$$|s^2 - 16| < 1$$
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What next?

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What next?

A clever trick

$$s \approx 4$$

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$$s \approx 4$$
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$$s \approx 4$$
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$$7 < |s+4| < 9$$

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- Know: $|s^2 - 16| = |s-4| \cdot |s+4| < 9|s-4|$

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$$|s^2 - 16| < 1$$
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- Know: $|s^2 - 16| = |s - 4| \cdot |s + 4| < 9|s - 4|$
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- Need: $9|s - 4| < 1$

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- Want: $|s^2 - 16| < 1$
- Need: $9|s-4| < 1 \Rightarrow |s-4| < 1/9$.

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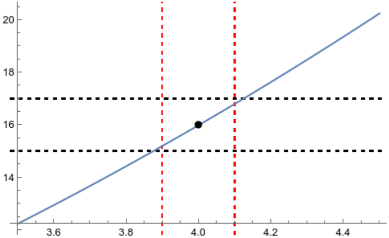
We can take $\delta = 1/9$.

Approximation Example 2 Part 3



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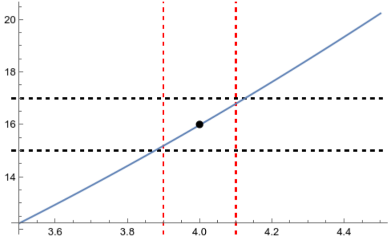

$1/9 \approx .111111 < 0.127017$ so this isn't optimal.

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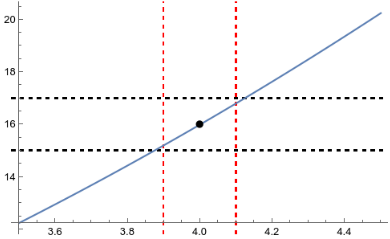

$1/9 \approx .111111 < 0.127017$ so this isn't optimal.
But we're approximating, so that's fine!

Heaviside Function

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Definition (Oliver Heaviside)

"Lightswitch function"

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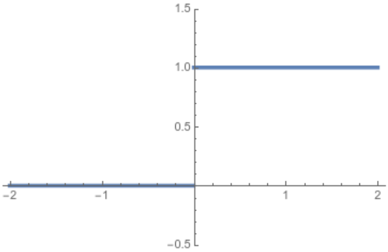

Dumb Questions about the Heaviside Function

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$\varepsilon = 2$

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- Want $I = 1 \pm 2$.

Dumb Questions about the Heaviside Function

$\varepsilon = 2$

- Want $l = 1 \pm 2$.
- l in $(-1, 3)$.

Dumb Questions about the Heaviside Function

$\varepsilon = 2$

- Want $I = 1 \pm 2$.
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- $\delta = 10$. Or 100. Or 10000000.

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$\varepsilon = .5$

- Want $I = 1 \pm .5$. I in $(.5, 1.5)$.

Dumb Questions about the Heaviside Function

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- l in $(-1, 3)$.
- $\delta = 10$. Or 100. Or 10000000.

$\varepsilon = .5$

- Want $l = 1 \pm .5$. l in $(.5, 1.5)$.
- Any $t \geq 0$.

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- What is δ? Depends on our target.

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$\varepsilon = 0.001$

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$\varepsilon = 0.001$

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- $l \in (0.999, 1.001)$.

Dumb Questions about the Heaviside Function

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- Any $t \geq 0$. Same as last time!

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$\varepsilon = 2$

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$\varepsilon = 0.001$

- Want $l = 1 \pm 0.001$.
- $l \in (0.999, 1.001)$.
- Any $t \geq 0$. Same as last time!

Subsection 3: Limits

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Question

What if we keep asking for more precision?

What happens as ε gets small?

Current and voltage

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What if we want $I = 5 \pm \varepsilon$?

$$|V/2 - 5| < \varepsilon$$

Current and voltage

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$$2|V/2 - 5| < 2\varepsilon$$

Current and voltage

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Need $\delta = 2\varepsilon$.

Current and voltage

$I = V/2$. When we wanted $I = 5 \pm 1/2$, we needed $V = 10 \pm 1$.

What if we want $I = 5 \pm \varepsilon$?

$$|V/2 - 5| < \varepsilon$$
$$2|V/2 - 5| < 2\varepsilon$$
$$|V - 10| 2\varepsilon$$

Need $\delta = 2\varepsilon$.

- If $\varepsilon = 0.5$ then $\delta = 1$

Current and voltage

$I = V/2$. When we wanted $I = 5 \pm 1/2$, we needed $V = 10 \pm 1$.

What if we want $I = 5 \pm \varepsilon$?

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- If $\varepsilon = 0.1$ then $\delta = 0.2$

Current and voltage

$I = V/2$. When we wanted $I = 5 \pm 1/2$, we needed $V = 10 \pm 1$.

What if we want $I = 5 \pm \varepsilon$?

$$|V/2 - 5| < \varepsilon$$
$$2|V/2 - 5| < 2\varepsilon$$
$$|V - 10|2\varepsilon$$

Need $\delta = 2\varepsilon$.

- If $\varepsilon = 0.5$ then $\delta = 1$
- If $\varepsilon = 0.1$ then $\delta = 0.2$
- If $\varepsilon = 0.001$ then $\delta = 0.002$.

Current and voltage

$I = V/2$. When we wanted $I = 5 \pm 1/2$, we needed $V = 10 \pm 1$.

What if we want $I = 5 \pm \varepsilon$?

$$|V/2 - 5| < \varepsilon$$
$$2|V/2 - 5| < 2\varepsilon$$
$$|V - 10| 2\varepsilon$$

Need $\delta = 2\varepsilon$.

- If $\varepsilon = 0.5$ then $\delta = 1$
- If $\varepsilon = 0.1$ then $\delta = 0.2$
- If $\varepsilon = 0.001$ then $\delta = 0.002$.

For *any* $\varepsilon > 0$, we can find a δ that will work.

Square area

$A = s^2$. When we wanted $A = 16 \pm 1$, we needed $s = 4 \pm 1/9$.

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$$\begin{aligned} |s^2 - 16| &= |s-4| \cdot |s+4| \\ &< 9|s-4| < \varepsilon \end{aligned}$$

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Need $\delta = \varepsilon/9$.

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What if we want $A = 16 \pm \varepsilon$?

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Need $\delta = \varepsilon/9$.

- If $\varepsilon = 1$ then $\delta = 1/9$

Square area

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$$\begin{aligned} |s^2 - 16| &= |s-4| \cdot |s+4| \\ &< 9|s-4| < \varepsilon \\ |s-4| &< \varepsilon/9 \end{aligned}$$

Need $\delta = \varepsilon/9$.

- If $\varepsilon = 1$ then $\delta = 1/9$
- If $\varepsilon = 0.1$ then $\delta = 1/90$

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What if we want $A = 16 \pm \varepsilon$?

$$\begin{aligned} |s^2 - 16| &= |s-4| \cdot |s+4| \\ &< 9|s-4| < \varepsilon \\ |s-4| &< \varepsilon/9 \end{aligned}$$

Need $\delta = \varepsilon/9$.

- If $\varepsilon = 1$ then $\delta = 1/9$
- If $\varepsilon = 0.1$ then $\delta = 1/90$
- If $\varepsilon = 0.001$ then $\delta = 1/9000$.

Square area

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Again, we can find a δ for any $\varepsilon > 0$

Definition of a limit

Definition

Suppose a is a real number, and f is a function defined on some open interval containing a, except possibly for at a. We say the *limit* of $f(x)$ as x approaches a is L, and write

$$\lim_{x \to a} f(x) = L,$$

if for every real number $\varepsilon > 0$ there is a real number $\delta > 0$ such that whenever $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

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This looks scary, but it's exactly what we've been doing already.

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If $f(x) = x/2$, prove $\lim_{x \to 10} f(x) = 5$.

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Proof.

Let $\varepsilon > 0$, and set $\delta = 2\varepsilon$. Then if $0 < |x - 10| < \delta$, we have

$$\begin{aligned} |f(x) - 5| = |x/2 - 5| &= \frac{1}{2}|x - 10| \\ &< \frac{1}{2}\delta = \frac{1}{2}(2\varepsilon) = \varepsilon. \end{aligned}$$

Thus $|f(x) - 5| < \varepsilon$, and so $\lim_{x \to 10} f(x) = 5$. □

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If $f(x) = 3x$ then prove $\lim_{x \to 1} f(x) = 3$.

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Proof.

Let $\varepsilon > 0$ and set $\delta = \underline{\varepsilon/3}$. Then if $0 < |x-1| < \delta$, we have

$$\begin{aligned} |f(x) - 3| &= |3x - 3| = 3|x-1| \\ &< 3\delta = \varepsilon. \end{aligned}$$

□

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Proof.

Let $\varepsilon > 0$ and set $\delta = \sqrt{\varepsilon}$. Then if $0 < |x - 0| < \delta$, we have

$$\begin{aligned} |f(x) - 0| &= |x^2 - 0| = |x|^2 \\ &< \delta^2 = \sqrt{\varepsilon}^2 = \varepsilon \end{aligned}$$

□

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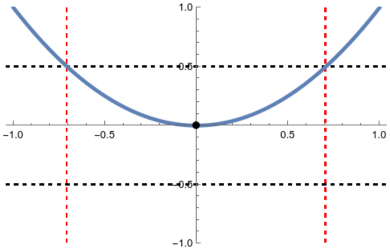

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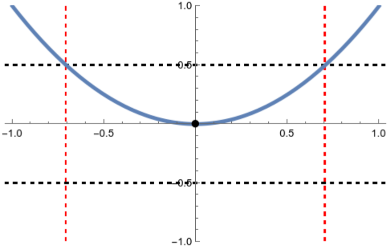

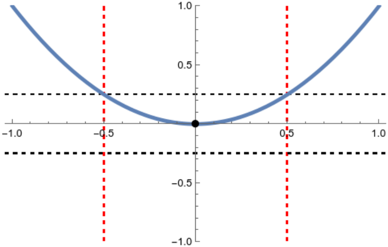

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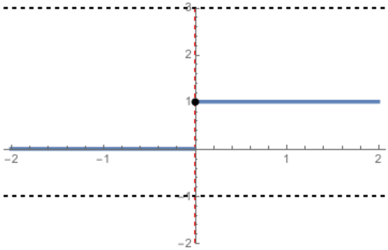
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2
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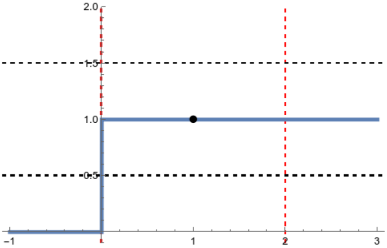
1
2


2.0
1.5
1.0
0.5
-1
1
2
3

How should we think about this?

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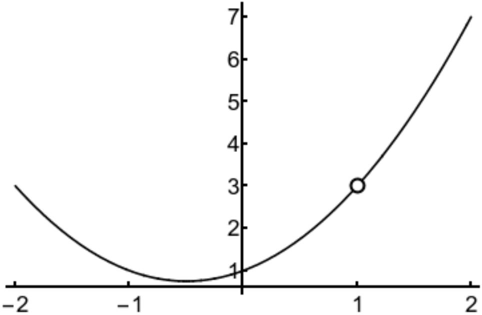
Definition (informal)

Suppose a is a real number, and f is a function which is defined for all x "near" the number a. We say "The *limit* of $f(x)$ as x approaches a is L," and we write

$$\lim_{x \to a} f(x) = L,$$

if we can make $f(x)$ get as close as we want to L by picking x that are very close to a.


7
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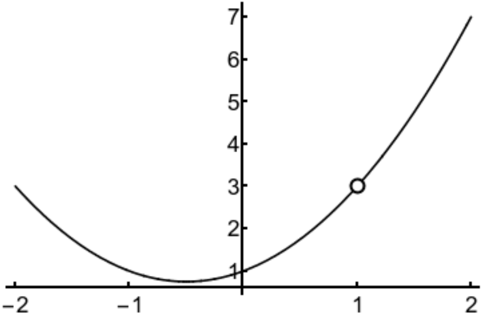

Lemma (Almost Identical Functions)

If $f(x) = g(x)$ on some open interval $(a-d, a+d)$ surrounding a, except possibly at a, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ whenever one limit exists.

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1. The function is defined at a; that is, a is in the domain of f.
2. $\lim_{x\to a} f(x)$ exists.
3. The two numbers are the same.

Ways to be discontinuous

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4. Some functions are just *really bad*. We'll see this with $\sin(1/x)$ when we study trigonometric functions in section **??**.

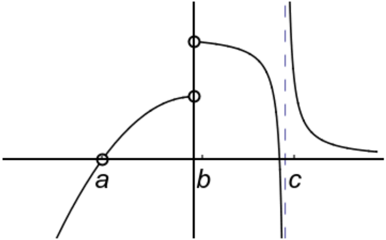

Figure: Left: a: removable discontinuity; b: jump discontinuity; c: infinite discontinuity.

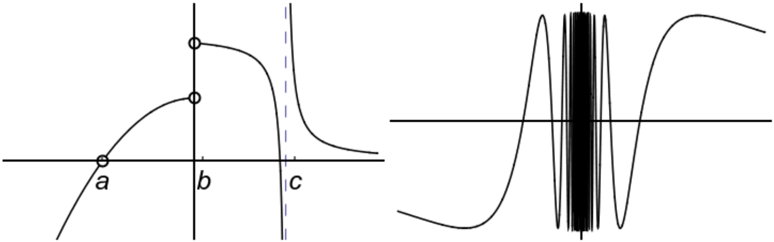

Figure: Left: a: removable discontinuity; b: jump discontinuity; c: infinite discontinuity. Right: a "bad" discontinuity in the function $\sin(1/x)$.

Some functions get even worse than that. My two favorite discontinuous functions are:

$$T(x) = \begin{cases} 1/q & x = p/q \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

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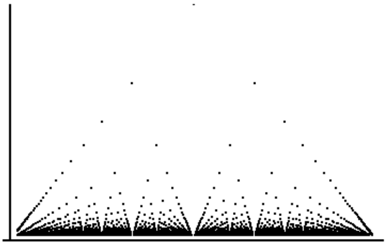

Figure: Left: $T(x)$ is really discontinuous.

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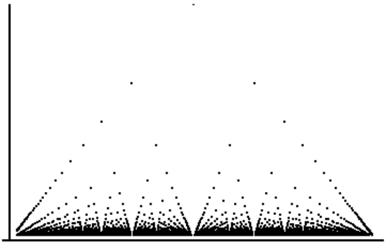
$$T(x) = \begin{cases} 1/q & x = p/q \text{ rational} \\ 0 & x \text{ irrational} \end{cases} \qquad \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$



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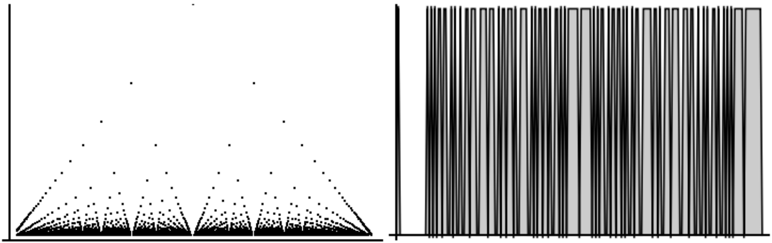

Figure: Left: $T(x)$ is really discontinuous. Right: $\chi_{\mathbb{Q}}(x)$ is really really discontinuous

Theorem (Intermediate Value Theorem)

Suppose f is continuous (and defined!) on the closed interval $[a, b]$ and y is any number between $f(a)$ and $f(b)$. Then there is a c in (a, b) with $f(c) = y$.