Math 1231 Midterm Solutions

Instructor: Jay Daigle

Name:

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Problem 1 (M3).

(a) Give a formula for the linear approximation of the function $f(x) = \frac{x+2}{x-1}$ near the point $a = 2$. Use your formula to estimate $f(2.2)$.

Solution:

$$\begin{aligned} f'(x) &= \frac{x-1-(x+2)}{(x-1)^2} \\ f'(2) &= \frac{-3}{1} = -3 \\ f(2) &= \frac{4}{1} = 4 \\ f(x) &\approx 4 - 3(x-2) \\ f(2.2) &\approx 4 - 3(2.2-2) = 4 - 3(.2) = 3.6. \end{aligned}$$

(b) Find a tangent line to the curve given by $y^3 + xy + \frac{x}{y} = 5$ at the point $(2, 1)$.

Solution: We use implicit differentiation, and find that

$$\begin{aligned} 3y^2y' + y + xy' + \frac{1}{y} - \frac{x}{y^2}y' &= 0 \\ 3y' + 1 + 2y' + 1 - 2y' &= 0 \\ 3y' &= -2 \\ y' &= -2/3 \end{aligned}$$

and so the equation of our tangent line is

$$y - 1 = \frac{-2}{3}(x-2).$$

Alternatively we could compute

$$\begin{aligned} 3y^2y' + y + xy' + \frac{1}{y} - \frac{x}{y^2}y' &= 0 \\ \left(3y^2 + x - \frac{x}{y^2}\right)y' &= -y - \frac{1}{y} \\ y' &= \frac{-y - 1/y}{3y^2 + x - \frac{x}{y^2}} \end{aligned}$$

so at the point $(2, 1)$ we get

$$y' = \frac{-2}{3+2-2} = \frac{-2}{3}$$

and so again the equation of our tangent line is

$$y - 1 = \frac{-2}{3}(x-2).$$

Problem 2 (M4).

(a) Find and classify all the critical points of $f(x) = x^4 + 3x^3 + x^2 - 3$.

Solution: We compute

$$f'(x) = 4x^3 + 9x^2 + 2x = x(4x^2 + 9x + 2) = x(4x + 1)(x + 2).$$

This is equal to zero when $x = -2, -1/4, 0$ and is never undefined, so the three critical points are $x = -2, -1/4, 0$.

We have two options here. We can use the second derivative test:

$$\begin{aligned} f''(x) &= 12x^2 + 18x + 2 \\ f''(-2) &= 48 - 36 + 2 = 14 > 0 \\ f''(-1/4) &= \frac{3}{4} - \frac{9}{2} + 2 = \frac{-7}{4} < 0 \\ f''(0) &= 2 > 0 \end{aligned}$$

so f has a local maximum at $x = -1/4$ and has local minima at $x = -2$ and $x = 0$.

Alternatively we can make a chart:

	x	$4x+1$	$x+2$	$f'(x)$
$x < -2$	$-$	$-$	$-$	$-$
$-2 < x < -1/4$	$-$	$-$	$+$	$+$
$-1/4 < x < 0$	$-$	$+$	$+$	$-$
$0 < x$	$+$	$+$	$+$	$+$

Thus f has a local maximum at $-1/4$ where the derivative switches from positive to negative, and has local minima at -2 and 0 where the derivative switches from negative to positive.

(b) Find the absolute extrema of $g(x) = \dfrac{2x - 2}{x^2 - 2x + 2}$ on $[0, 3]$, and justify your claim that these are the absolute extrema.

Solution: This function is defined everywhere and thus continuous everywhere. Thus it is continuous on $[0, 3]$, and so by the Extreme Value Theorem it achieves a maximum and a minimum.

We compute

$$\begin{aligned} g'(x) &= \frac{2(x^2 - 2x + 2) - (2x - 2)(2x - 2)}{(x^2 - 2x + 2)^2} \\ &= \frac{2x^2 - 4x + 4 - (4x^2 - 8x + 4)}{(x^2 - 2x + 2)^2} \\ &= \frac{-2x^2 + 4x}{(x^2 - 2x + 2)^2} \\ &= \frac{2(2 - x)x}{(x^2 - 2x + 2)^2} \end{aligned}$$

This is never undefined, and it's 0 when $x = 0, 2$. So then we have

$$\begin{aligned} g(0) &= \frac{-2}{2} = -1 \\ g(2) &= \frac{2}{2} = 1 \\ g(3) &= \frac{4}{5}. \end{aligned}$$

Thus the minimum value is -1 and the maximum value is 1.

Problem 3 (S3)**.** Suppose we have a gas in a compressible tank, and the gas has pressure $P(V) = \frac{4}{V}$ pascals when the tank has a volume of V liters.

(a) What are the units of $P'(V)$? What does it represent physically? If $P'(V)$ is a negative number, what does that tell us?

Solution: $P'(V)$ has units of pascals per liter. The derivative gives the rate at which the pressure changes when the volume of the tank increases. If $P'(V)$ is a negative number, that means making the tank bigger will make the pressure smaller.

(b) Calculate $P'(2)$. What does that tell you physically? What measurements could you do to check your calculation?

Solution: $P'(V) = \frac{-4}{V^2}$ so $P'(2) = \frac{-4}{4} = -1$. This tells us that if we increase the volume of the tank by 1 liter, the pressure should drop by about 1 Pascal. We could measure the pressure at 2 liters and at 3 liters and see if the difference is about 1 Pascal.

Problem 4 (S4)**.** A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 feet per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence that clearly and directly answers the question.

Solution: We know how quickly the bottom is sliding away from the wall, and we want to know how fast the top is sliding down. The ladder forms a triangle against the wall, and we want to relate the lengths of two sides, so we use the Pythagorean theorem $h^2 + b^2 = c^2$.

We know $b = 6$ and $c = 10$; by the Pythagorean theorem, we know that $h = 8$. Further we know that $b' = 2$ and $c' = 0$. So taking a derivative gives

$$\begin{aligned} 2hh' + 2bb' &= 2cc' \\ 2 \cdot 8h' + 2 \cdot 6 \cdot 2 &= 0 \\ 16h' &= -24 \\ h' &= -3/2. \end{aligned}$$

Thus the top of the ladder is sliding down the wall at 1.5 feet per second.

Problem 5 (S5)**.** Let $f(x) = (x^2-1)^{2/3}$. We compute that $f'(x) = \dfrac{4x}{3\sqrt[3]{x^2-1}}$ and $f''(x) = \dfrac{4(x^2-3)}{9(x^2-1)^{4/3}}$. Sketch a graph of f.

Your answer should discuss the domain, roots, limits at infinity, critical points and values, intervals of increase and decrease, and potential points of inflection, and concavity.

Solution: The function is defined everywhere. We see there are roots at $x = \pm 1$, and $\lim_{x\to\pm\infty} f(x) = +\infty$.

We see that $f'(x)$ is undefined at $x = \pm 1$, and is zero when $x = 0$. So our critical points occur at $-1, 0, 1$. We calculate $f(0) = 1$, and $f(-1) = f(1) = 0$. By making a chart, we get

	$4x$	$\frac{1}{3\sqrt[3]{x^2-1}}$	$f'(x)$
$x < -1$	$-$	$+$	$-$
$-1 < x < 0$	$-$	$-$	$+$
$0 < x < 1$	$+$	$-$	$-$
$1 < x$	$+$	$+$	$+$

so f is decreasing on $(-\infty, -1)$ and $(0, 1)$ and it's increasing on $(-1, 0)$ and $(1, +\infty)$.

Alternatively, we could make the chart like this:

	$4x$	$\frac{1}{3\sqrt[3]{x-1}}$	$\frac{1}{\sqrt[3]{x+1}}$	$f'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$1 < x$	$+$	$+$	$+$	$+$

which is probably the cleaner way to do it but maybe less obvious. (We get the same answer either way.)

The second derivative is undefined at ± 1 and is zero at $\pm\sqrt{3}$. We compute $f(\sqrt{3}) = 2^{2/3} = f(-\sqrt{3})$, and still have $f(\pm 1) = 0$. We can again make a chart:

	$4(x^2-3)$	$\frac{1}{9(x^2-1)^{4/3}}$	$f'(x)$
$x < -\sqrt{3}$	$+$	$+$	$+$
$-\sqrt{3} < x < -1$	$-$	$+$	$-$
$-1 < x < 1$	$-$	$+$	$-$
$1 < x < \sqrt{3}$	$-$	$+$	$-$
$\sqrt{3} < x$	$+$	$+$	$+$

so the function is concave up for $x < -\sqrt{3}$ or $x > \sqrt{3}$, and it's concave down for $-\sqrt{3} < x < \sqrt{3}$. Again, we could also write the chart as

	$4(x-\sqrt{3})$	$x+\sqrt{3}$	$\frac{1}{9(x^2-1)^{4/3}}$	$f'(x)$
$x < -\sqrt{3}$	$-$	$-$	$+$	$+$
$-\sqrt{3} < x < -1$	$-$	$+$	$+$	$-$
$-1 < x < 1$	$-$	$+$	$+$	$-$
$1 < x < \sqrt{3}$	$-$	$+$	$+$	$-$
$\sqrt{3} < x$	$+$	$+$	$+$	$+$