Math 1231 Practice Final Solutions

Instructor: Jay Daigle

Problem 1 (M1). (a) Compute $\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25}$

Solution:

$$\lim_{x\to 25} \frac{\sqrt{x}-5}{x-25} = \lim_{x\to 25} \frac{x-25}{(x-25)(\sqrt{x}+5)} = \lim_{x\to 25} \frac{1}{\sqrt{x}+5} = \frac{1}{10}.$$

(b) Compute $\lim_{x\to -\infty} \frac{3x}{\sqrt{4x^2+3}}$

Solution:

$$\begin{aligned}\lim_{x\to -\infty} \frac{3x}{\sqrt{4x^2+3}} &= \lim_{x\to -\infty} \frac{3x/x}{\sqrt{4x^2+3}/(-\sqrt{x^2})} \\ &= \lim_{x\to -\infty} \frac{3}{-\sqrt{4+3/x^2}} = \frac{-3}{2}.\end{aligned}$$

Problem 2 (M2). (a) Find $\frac{d}{dx}\sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4} \cdot \frac{(3x^2-\sin(x^2)2x)(\sin(x^3)+1)-\cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

(b) Find a formula for y' in terms of x and y if $x^8+x^4+y^4+y^6=1$.

Solution:

$$\begin{aligned}8x^7+4x^3+4y^3\frac{dy}{dx}+6y^5\frac{dy}{dx} &= 0 \\ 8x^7+4x^3 &= -(4y^3+6y^5)\frac{dy}{dx} \\ -\frac{4x^7+2x^3}{2y^3+3y^5} &= \frac{dy}{dx}.\end{aligned}$$

Problem 3 (M3). (a) If $f(x)=\sqrt{x}+\tan(\pi x)$, use a linear approximation centered at 4 to estimate $f(4.1)$.

Solution: We have $f'(x) = \frac{1}{2\sqrt{x}} + \pi\sec^2(\pi x)$ so $f'(4) = \frac{1}{4}+\pi$. Then

$$\begin{aligned}f(x) &\approx f(4)+f'(4)(x-4) = 2+0+(\pi+1/4)(x-4) \\ f(4.1) &\approx 2+\frac{\pi}{10}+\frac{1}{40} = \frac{81}{40}+\frac{\pi}{10}.\end{aligned}$$

(b) A curve is defined by the equation $x^4-2x^2y^2+y^4=16$. $(\sqrt{5},1)$. What is the equation of the tangent line to the curve at the point $(\sqrt{5},1)$?

Solution: If we plug in $\sqrt{5}$ for x and 1 for y we get $25 - 2 \cdot 5 \cdot 1 + 1 = 16$, so the point $(\sqrt{5}, 1)$ is on the curve.

To find the tangent line, we use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$4x^3 - 4xy^2 = 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

$$\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} = \frac{dy}{dx}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5}\cdot 1^2}{4\sqrt{5}^2 \cdot 1 - 4 \cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \sqrt{5}(x - \sqrt{5}).$$

Problem 4 (M4). (a) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x-4)(x-1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$f(0) = 7$$
$$f(1) = 14$$
$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$
$$f(5) = 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

(b) Find and classify the critical points of $g(x) = \dfrac{2x-1}{x^2+2}$.

Solution: We have

$$g'(x) = \frac{2(x^2+2) - 2x(2x-1)}{(x^2+2)^2} = \frac{-2x^2 + 2x + 4}{(x^2+2)^2}$$
$$= -2\frac{x^2 - x - 2}{(x^2+2)^2} = -2\frac{(x-2)(x+1)}{(x^2+2)^2}$$

so the critical points are 2 and -1. (The derivative is defined everywhere).

To classify these critical points we need to use either the first or second derivative test. I think the first derivative test looks easier here, purely because I don't want to compute the second derivative. I get the table

	$x-2$	$x+1$	$\frac{-2}{(x^2+2)^2}$	$g'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 2$	$-$	$+$	$-$	$+$
$2 < x$	$+$	$+$	$-$	$-$

Thus we see that there is a relative minimum at -1 and a relative maximum at 2.

But we could use the second derivative test if we really wanted to. We compute

$$g''(x) = -2\frac{(2x-1)(x^2+2)^2 - 2(x^2+2)2x(x^2-x-2)}{(x^2+2)^4}$$

$$g''(-1) = -2\frac{(-3)(3)^2 - 2(3)(-2)(0)}{3^4} = \frac{-2\cdot(-27)}{3^4} = 2/3 > 0$$

$$g''(2) = -2\frac{3(6)^2 - 2(6)4(0)}{6^4} = \frac{-1}{6} < 0.$$

Thus $g''(-1) > 0$ so g has a minimum at -1; and $g''(2) < 0$ so g has a maximum at 2.

Problem 5 (M5). (a) Compute $\int \sin^4(t)\cos(t)\,dt$

Solution: We can take $u = \sin(t)$, then we have $du = \cos(t)\,dt$ so we are computing

$$\int u^4\,du = \frac{1}{5}u^5 + C = \frac{\sin^5(t)}{5} + C.$$

(b) **By explicitly changing the bounds of the integral**, compute $\int_0^4 x^3\sqrt{9+x^2}\,dx$.

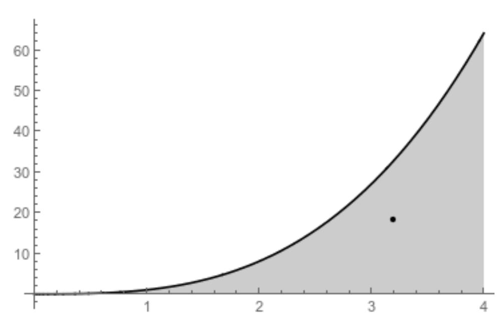
Solution: Take $u = 9+x^2$ so that $x^2 = u-9$ and $dx = du/2x$. Then $u(0) = 9$ and $u(4) = 25$ and we have

$$\int_0^4 x^3\sqrt{9+x^2}\,dx = \int_9^{25} x^3\sqrt{u}\frac{du}{2x} = \int_9^{25} \frac{1}{2}(u-9)\sqrt{u}\,du$$
$$= \frac{1}{2}\int_9^{25} u^{3/2} - 9u^{1/2}\,du$$
$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - 6u^{3/2}\right)\Big|_9^{25} = \frac{1}{2}(1250 - 750 - (486/5 - 162))$$
$$= \frac{1}{2}(662 - 486/5) = \frac{3310-486}{10} = \frac{2824}{10} = 1412/5.$$

Problem 6 (M6). (a) What is the x-coordinate of the center of mass of the region bounded by $y = x^3$, $y = 0$, and $x = 4$?

Solution: The area of the region is $\int_0^4 x^3\,dx = \frac{x^4}{4}|_0^4 = 64$. The x coordinate of the center of mass is

$$\overline{x} = \frac{1}{A}\int_0^4 x\cdot x^3\,dx = \frac{1}{64}\cdot\frac{x^5}{5}|_0^4 = \frac{16}{5}.$$



(b) A spring with natural length of 8 inches takes 6 pounds of force to stretch to 10 inches. Set up (but do not evaluate) an integral to compute the work done by stretching the spring from 12 inches to 16 inches. What units will this integral output?

Solution: We know that $F(x) = k(x - x_0)$. We have $x_0 = 8$, and then $6 = F(10) = k(10 - 8) = 2k$ so that $k = 3$lbs/in. Thus the work done will be

$$\int_{12}^{16} 3(x-8)\,dx.$$

This will produce an answer in pound-inches.

Problem 7 (S6)**.** Suppose you are running a toy shop. It costs $C(x) = 200 + 10x$ dollars to produce x toys in a day, and you make a revenue of $R(x) = 26x - .2x^2$ dollars if you sell x toys in a day. How many toys should you produce per day to maximize your profit? Make sure you give a complete sentence for your answer.

Solution: Our profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) = (26x - .2x^2) - (200 + 10x) \\ &= 16x - .2x^2 - 200 \\ P'(x) &= 16 - .4x \end{aligned}$$

To find a critical point, We set $16 - .4x = 0$, which gives $16 = .4x$ or $40 = x$.

Since this is the only critical point, we expect it to be our maximum. We can provide evidence for this by looking at either the first or second derivative. We see that $P'(x) > 0$ for $x < 40$ and $P'(x) < 0$ for $x > 40$, which implies that P has a local maximum at 40. Further, since the function is *always* increasing for $x < 40$ and *always* decreasing for $x > 40$ this must be a global maximum.

We can also look at the second derivative and see that $P''(x) = -.4 < 0$. Thus the function is concave down, and any local extremum must be a critical point.

Therefore: we should make 40 toys per day.

Problem 8 (S7)**.** Using **only the definition of Riemann sum** and your knowledge of limits, compute the exact area under the curve $x^2 + x^3$ between $x = 1$ and $x = 3$.

Solution: We compute

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \frac{2}{n} f\left(1 + \frac{2i}{n}\right) = \frac{2}{n} \sum_{i=1}^{n} (1 + 2i/n)^2 + (1 + 2i/n)^3 \\ &= \frac{2}{n} \sum_{i=1}^{n} (1 + 4i/n + 4i^2/n^2) + (1 + 6i/n + 12i^2/n^2 + 8i^3/n^3) \\ &= \frac{2}{n} \sum_{i=1}^{n} 2 + 10i/n + 16i^2/n^2 + 8i^3/n^3 \\ &= \frac{4}{n} \sum_{i=1}^{n} 1 + \frac{20}{n^2} \sum_{i=1}^{n} i + \frac{32}{n^3} \sum_{i=1}^{n} i^2 + \frac{16}{n^4} \sum_{i=1}^{n} i^3 \\ &= \frac{4}{n} \cdot n + \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ \lim_{n \to +\infty} R_n &= \lim_{n \to +\infty} \frac{4}{n} \cdot n + \frac{20}{n^2} \cdot \frac{n(n+1)}{2} + \frac{32}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4} \cdot \frac{n^2(n+1)^2}{4} \\ &= 4 + 10 + \frac{32}{3} + 4 = \frac{86}{3}. \end{aligned}$$

These questions are optional. Only attempt these if you have finished all the major topics and topic S6-7, and want to improve your mastery scores.

Problem 9 (S1)**.** Suppose $f(x) = x^2 + 3$, and we want an output of approximately 19. If we want our input to be positive, what input a should we aim for? Find a δ so that if our input is $a \pm \delta$ then our output will be 19 ± 1. Explain how you found this δ and why it should give us what we want.

Solution: We want an input of about $a = 4$. We want

$$\begin{aligned} |x^2 + 3 - 19| &< 1 \\ |x^2 - 16| &< 1 \\ |x - 4||x + 4| &< 1. \end{aligned}$$

Since $x \approx 4$ we can assume $|x + 4| \approx 8$ and in particular $|x + 4| < 9$. Then we want

$$\begin{aligned} |x - 4||x + 4| < |x - 4| \cdot 9 &< 1 \\ |x - 4| &< 1/9 \end{aligned}$$

So we take $\delta = 1/9$.

Problem 10 (S2)**. Directly from the definition**, compute $f'(1)$ where $f(x) = \sqrt{x + 3}$.

Solution:

$$\begin{aligned} f'(1) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{4 + h} - \sqrt{4}}{h} \\ &= \lim_{h \to 0} \frac{(4 + h) - 4}{h(\sqrt{4 + h} + \sqrt{4})} = \lim_{h \to 0} \frac{1}{\sqrt{4 + h} + 2} = \frac{1}{4}. \end{aligned}$$

Problem 11 (S3)**.** Suppose that if a car travels at v miles per hour then its fuel efficiency is $F(v) = 8 + 1.3v - .015v^2$ miles per gallon.

(i) What does the derivative $F'(v)$ represent, and what are its units?

Solution: The derivative $F'(v)$ is the rate at which fuel efficiency increases as your speed increases. The units are miles per gallon per mile per hour, which winds up working out to hours per gallon. (This is a little weird, but it actually makes sense: it's something like how many hours you save by burning an extra gallon of fuel).

(ii) Compute $F'(60)$. What does this tell you?

Solution: $F'(v) = 1.3 - .03v$ hours per gallon so $F'(60) = 1.3 - 1.8 = -.5$ hours per gallon. This tells us that if we are going sixty miles per hour, then increasing our speed by one mile per hour will reduce our gas milage by half a mile per gallon.

Problem 12 (S4)**.** A cone with height h and base radius r has volume $\frac{1}{3}\pi r^2 h$. Suppose we have an inverted conical water tank, of height 4m and radius 6m. Water is leaking out of a small hole at the bottom of the tank. If the current water level is 2m and the water level is dropping at $\frac{1}{9\pi}$ meters per minute, what volume of water leaks out every minute?

Solution: We have $V = \frac{1}{3}\pi r^2 h$ and $r = 3h/2$, and thus

$$\begin{aligned} V &= \frac{1}{3}\pi(\frac{3h}{2})^2 h = \frac{3}{4}\pi h^3 \\ V' &= \frac{9}{4}\pi h^2 h' \\ V' &= \frac{9}{4}\pi(2)^2 \frac{-1}{9\pi} = 1 \end{aligned}$$

So one cubic meter of water is leaking out every minute.

Problem 13 (S5)**.** Let $j(x) = x^4 - 14x^2 + 24x + 6$. We can compute $j'(x) = 4(x + 3)(x - 1)(x - 2)$ and $j''(x) = 4(3x^2 - 7)$. Sketch a graph of j.

Your answer should discuss the domain, asymptotes, limits at infinity, critical points and values, intervals of increase and decrease, and concavity.

Solution: The domain of j is all reals. I'm not going to worry about finding roots now , and there are no obvious symmetries. It's a polynomial of even degree, so it's easy to see that $\lim_{x\to\pm\infty} j(x) = +\infty$.

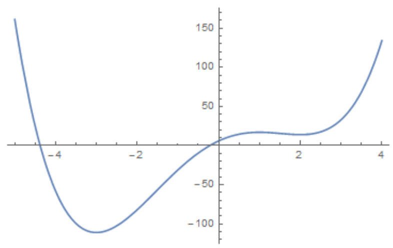
The function j is defined everywhere and is zero at three points. Thus j has three critical points, at $-3, 1, 2$. We compute j at these critical points: $j(-3) = 81 - 126 - 72 + 6 = -111, j(1) = 1 - 14 + 24 + 6 = 17, j(2) = 14$.

We can make a chart to determine when j increases or decreases:

	$(x+3)$	$(x-1)$	$(x-2)$	$j'(x)$
$x < -3$	$-$	$-$	$-$	$-$
$-3 < x < 1$	$+$	$-$	$-$	$+$
$1 < x < 2$	$+$	$+$	$-$	$-$
$2 < x$	$+$	$+$	$+$	$+$

So j is increasing between -3 and 1 and when bigger than 2, and j is decreasing when smaller than -3 or between 1 and 2. This implies that j has a relative minimum (of -111) at -3, a relative maximum (of 17) at 1, and a relative minimum of 14 at 2.

$j''(x) = 4(3x^2 - 7)$ is defined eveyrwhere, and is zero when $x^2 = 7/3$, when $x = \pm\sqrt{7/3}$. $j''(x)$ is positive when $|x| > \sqrt{7/3}$ and negative when $|x| < \sqrt{7/3}$.



Graph of $j(x)$