Math 1231 Practice Midterm Solutions

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Problem 1 (M1)**.** Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(b)

$$\lim_{x\to -\infty} \frac{3x^3 + \sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned} \lim_{x\to -\infty} \frac{3x^3 + \sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3 + \sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6}) + x/x^3} \\ &= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\ &= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1. \end{aligned}$$

(c)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(d)

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 2 (M2)**.** (a) Explicitly justifying each step and naming each derivative rule you use, compute $\frac{d}{dx}\frac{\sin(x)+1}{2x^2-5}$.

Solution:

$$\begin{aligned}
\frac{d}{dx}\frac{\sin(x)+1}{2x^2-5} &= \frac{(\sin(x)+1)'(2x^2-5)-(2x^2-5)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Quotient rule}\\
&= \frac{\big((\sin(x))'+(1)'\big)(2x^2-5)-\big((2x^2)'-(5)'\big)(\sin(x)+1)}{(2x^2-5)^2} && \text{Sum Rule}\\
&= \frac{(\sin(x))'(2x^2-5)-(2x^2)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Constants rule}\\
&= \frac{\cos(x)(2x^2-5)-(2x^2)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Derivative of sine}\\
&= \frac{\cos(x)(2x^2-5)-2(x^2)'(\sin(x)+1)}{(2x^2-5)^2} && \text{Scalar Products}\\
&= \frac{\cos(x)(2x^2-5)-2(2x)(\sin(x)+1)}{(2x^2-5)^2} && \text{Power Rule.}
\end{aligned}$$

(b) Using any methods we've developed in class, compute the derivative of $g(x) = \tan(x)\sqrt[3]{2x+1}$.

Solution:

$$g'(x) = \sec^2(x)\sqrt[3]{2x+1} + \tan(x)\frac{1}{3}(2x+1)^{-2/3}\cdot 2$$

(c) Using any methods we've developed in class, compute the derivative of $h(x) = \sec^3\left(\frac{x^3+1}{x-1}\right)$.

Solution:

$$h'(x) = 3\sec^2\left(\frac{x^3+1}{x-1}\right)\sec\left(\frac{x^3+1}{x-1}\right)\tan\left(\frac{x^3+1}{x-1}\right)\frac{3x^2(x-1)-(x^3+1)}{(x-1)^2}$$

Problem 3 (S1)**.**

Suppose $f(x) = x^2 - 6x$, and we want an output of approximately -9. What input a should we aim for? Find a δ so that if our input is $a \pm \delta$ then our output will be -9 ± 2. Justify your answer.

Solution: We want an input of about $a = 3$. Our output error will be $|x^2 - 6x + 9| = |x-3|^2$. We want this to be less than 2, so we need

$$\begin{aligned}
|x-3|^2 &< 2\\
|x-3| &< \sqrt{2},
\end{aligned}$$

so we can take $\delta = \sqrt{2}$.

Problem 4 (S2)**. Directly from the definition of derivative,** compute the derivative of $f(x) = x^2 + \sqrt{x}$ at $a = 2$.

Solution:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2+\sqrt{2+h}-2^2-\sqrt{2}}{h} \\
&= \left(\lim_{h\to 0} \frac{4h+h^2}{h}\right) + \left(\lim_{h\to 0} \frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right) \\
&= \left(\lim_{h\to 0} 4+h\right) + \left(\lim_{h\to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}\right) \\
&= 4 + \frac{1}{2\sqrt{2}}.
\end{aligned}$$