2 Derivatives

2.1 Linear Approximation

In the last section we talked about continuous functions as functions that we could approximate. We know that $\sqrt{5}$ is about 2, and 3.1^3 is about 27. In this section we want to be a bit more precise than that. Most of you told me not only that $\sqrt{5}$ is "about 2", but it's a bit *more* than 2. We want to find a way to estimate that bit more.

We need to use a more complicated formula. But we want to keep the amount of complexity under control. So we want to use a simple function to approximate $f(x)$. The simplest possible function is a constant function; and that's exactly what we used last section. (3.1^3 is about 27, and 3.01^3 is about 27, and 3.2^3 is about 27.) If a is a fixed number then $f(a)$ is a constant, and thus $f(x) \approx f(a)$ approximates f with a constant function.

The next most complex function, as we usually think of it, is a linear function. So we want to approximate f with a linear function. There are a few ways we can write the equation for a line, depending on what information we already know:

$$y = mx + b \qquad \text{Slope-Intercept Formula}$$

$$y - y_0 = m(x - x_0) \qquad \text{Point-Slope Formula}$$

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \qquad \text{Two Points Formula}$$

The most common and popular is the slope-intercept formula, which is great for *computing* things; but to write down the equation, you need to know the slope m, and also the y-intercept b. For our approximations we won't generally know this.

The two points formula also isn't terribly useful for us. We know one point: since we're approximating a function f near a, we know it goes through the point $(a, f(a))$. But if we knew the value at other points, we wouldn't need to approximate! (The approximation $f(x) - f(a) \approx \frac{f(x)-f(a)}{(x-a)}(x - a)$ is true, but is kind of vacuous and tautological; it doesn't actually help us).

But the point-slope formula can get us somewhere. We already have a point, so we just need to find the slope. We'll see how to do that soon, but for now we'll just give the slope a name: if we're taking a linear approximation to a function $f(x)$ near a point a, then we will denote the slope $f'(a)$. This tells us, essentially, how much we care about the distance between x and a. When this is small, then $f(x)$ is close to $f(a)$; when $f'(a)$ is large, then $f(x)$ moves away from $f(a)$ pretty quickly.

The equation for our linear approximation is

$$f(x) \approx f'(a)(x-a) + f(a) \tag{1}$$

This is the most important formula in the entire course; essentially everything we do for the next two months will refer back to this approximation in some way.

Example 2.1. We earlier said that $\sqrt{5} \approx \sqrt{4} = 2$. We can see that in fact $\sqrt{5}$ should be a little bigger than 2. But how much better?

A linear approximation would tell us that $\sqrt{5} \approx 2 + f'(2)(5-4)$. That is, we know that $\sqrt{5}$ is a bit bigger than two—and it's a bit bigger by the amount of this mysterious $f'(2)$ slope. We'll see how to compute this later, but for right now I'll tell you that $f'(2) = \frac{1}{4}$. Then we get that $\sqrt{5} \approx 2 + \frac{1}{4}(5-4) = 9/4 = 2.25$.

From this we can make other estimates. For instance, we have that $\sqrt{4.5} \approx 2 + \frac{1}{4}(4.5-4) = 17/8$, and $\sqrt{6} \approx 2 + \frac{1}{4}(6-4) = 5/2$.

We can go in the other direction as well. We estimate that $\sqrt{3} \approx 2 + \frac{1}{4}(3-4) = 7/4$. And $\sqrt{2} \approx 2 + \frac{1}{4}(2-4) = 3/2$.

But notice: this gives us $\sqrt{1} \approx 2 + \frac{1}{4}(1-4) = 5/4$, which we know is wrong. And $\sqrt{9} \approx 2 + \frac{1}{4}(9-4) = 13/4$, which is also wrong. For that matter, we get $\sqrt{100} \approx 2 + \frac{1}{4}(100-4) = 26$, which is really wrong. What's going on here?

We can even graph the formula for this approximation: we have

$$\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$$

and if we graph $\sqrt{x}$ and $2 + \frac{1}{4}(x-4)$ on the same graph, we get figure 2.1: We may notice

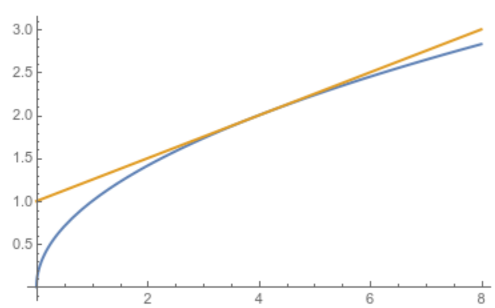


Figure 2.1: $\sqrt{x}$ (blue) and its linear approximation at 4 (yellow)

that the graph of our linear approximation is *tangent* to the graph of our function. We'll revisit this idea in section 2.8

A linear approximation is good when x is close to $a = 4$. As x gets further away from a, then our estimate for $f(x)$ gets further from $f(a)$; but in general we would also expect our estimate to get further from the correct answer. These techniques work best when x is very close to a.

(We're not yet ready to be precise about what "very close" means here).

Example 2.2. We've dressed this up in fancy language, but we engage in this sort of reasoning all the time. Suppose you are driving at 30 miles per hour. After an hour, you expect to have gone about thirty miles. After six minutes, you expect to have gone about three miles.

This is just a linear approximation. If $f(t)$ is our position as a function of time, our approximation is that we're moving 30 miles per hour, or half a mile per minute. Then we have $f(t) \approx 0 + \frac{1}{2}(t-0)$, and if we plug in $t = 6$ we have $f(6) \approx 0 + \frac{1}{2}(6-0) = 3$.

We'll revisit this idea in section 2.7.

2.2 The Derivative

We understand that we want to do linear approximation now. But without a way to actually find the slope $f'(a)$, it isn't terribly helpful.

So let's look at our formula from equation (3) again. We want to understand $f'(a)$, so we'll solve the equation for that:

$$\begin{aligned} f(x) &\approx f'(a)(x-a) + f(a) \\ f(x) - f(a) &\approx f'(a)(x-a) \\ \frac{f(x)-f(a)}{x-a} &\approx f'(a). \end{aligned}$$

Thus we get a new formula. This formula should also make sense to us. The slope $f'(a)$ tells us how different $f(x)$ is from $f(a)$, based on how x is different from a. This new, rearranged formula tells us that $f'(a)$ approximates the ratio of the change in $f(x)$ to the change in x, which we sometimes write as $\frac{\Delta f}{\Delta x}$. Thus it should tell us how much a change in the input value affects the output value—which is exactly the question we need to answer to write a linear approximation.

But we've also seen this formula somewhere else. In the two points formula for a line, the slope is $\frac{y_1-y_0}{x_1-x_0}$. if $y_1 = f(x_1) = f(x)$ and $y_0 = f(x_0) = f(a)$, then this is just the approximation we have for $f'(a)$. Thus we're saying that $f'(a)$ is approximately the slope of the line through the point $(a, f(a))$ that we know, and the point $(x, f(x))$ that we want. We'll explore this angle more in lab.

On its own, this still isn't helpful: we have an approximate formula for $f'(a)$, but it requires us to already know $f(x)$, which is what we started out wanting to compute. But one more step makes this actually useful.

Definition 2.3. Let f be a function defined near and at a point a. We say the *derivative* of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The second formula is just a change of variables from the first, setting $h = x - a$. It's not substantively any different, but it's sometimes easier to compute with.

We will also sometimes write $\frac{df}{dx}(a)$ for the derivative of f at a. The is called "Leibniz notation", as opposed to the "Newtonian notation" of $f'(a)$.

Thus the derivative is given by taking our approximate formula for $f'(a)$, and taking the limit as x and a get closer together. Our linear approximation is better when x and a are closer; so as x approaches a, the approximation becomes perfect, and we get an exact equation.

Remark 2.4. Note that we need *two* pieces of information here. You hand me a function f *and* a point a, and I tell you the derivative of f at a. We'll adopt different perspectives from time to time later on in the course.

Example 2.5. Let $f(x) = x^2 + 1$. Then

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^2 + 1 - 2^2 - 1}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = 4,$$

and more generally, for any number a we have

$$f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h \to 0} \frac{2ah + h^2}{h} = 2a.$$

Example 2.6. Let $f(x) = \sqrt{x}$. Then given a number a, we have

$$f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} = \lim_{h \to 0} \frac{(a+h) - a}{h(\sqrt{a+h} + \sqrt{a})} = \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

Note that f is defined at 0, and we have $f(0) = 0$. But by this computation we have $f'(0) = \frac{1}{2 \cdot 0}$ which is undefined. This isn't an artifact of the way we computed it; the limit in fact does not exist. Further, this isn't just becasue 0 is on the edge of the domain of f, as we shall see:

Example 2.7. Let $g(x) = \sqrt[3]{x}$. Then we can compute $g'(0)$ and we get

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = +\infty.$$

The cube root function g has no defined derivative at 0, even though the function is defined there. This brings us to a discussion of ways for a function to fail to be differentiable at a point. (There's always the catchall category of "the limit just doesn't exist," which we won't really discuss because there's not much to say about it).

1. **Vertical Tangent Line**

   Our first example of $g(x) = \sqrt[3]{x}$ is not differentiable at 0, and the limit

   $$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = +\infty.$$

   Graphically, the line tangent to g at 0 is completely vertical; the function is "increasing infinitely fast" at 0.

2. **Corner**

   Any function with a sharp corner at a point doesn't have a well-defined rate of change at that point; the change is instantaneous. For instance, if we let $a(x) = |x|$ be the absolute value function, then

   $$a'(x) = \lim_{h \to 0} \frac{a(x+h) - a(x)}{h}.$$

   To study piecewise functions we usually break them up and study each piece separately. If $x > 0$, then $a(x) = x$ and $a(x+h) = x + h$ for small h. We have

   $$a'(x) = \lim_{h \to 0} \frac{x + h - x}{h} = \lim_{h \to 0} 1 = 1.$$

   Conversely, if $x < 0$ then $a(x) = -x$ and $a(x+h) = -x - h$, and

   $$a'(x) = \lim_{h \to 0} \frac{-x - h + x}{h} = \lim_{h \to 0} -1 = -1.$$

   But if $x = 0$ then the left and right limits don't agree again: the right limit is 1 and the left limit is -1, so the limit does not exist. Thus we have

   $$a'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0. \end{cases}$$

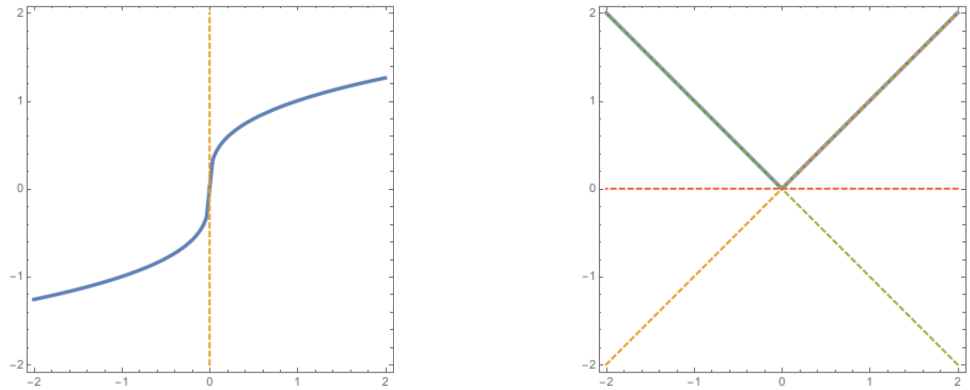


Figure 2.2: A vertical tangent line and a corner

3. **Cusp**

   Sometimes a function has a "cusp" at a point. This is a point where the tangent line is vertical, but depending on the side from which you approach, you can get a tangent line that goes up incredibly fast or one that goes down incredibly fast.

   Consider the funtion $f(x) = \sqrt[3]{x^2}$. We have

   $$f'(0) = \lim_{h \to 0} \frac{\sqrt[3]{h^2} - \sqrt[3]{0}}{h} = \lim_{h \to 0} \frac{h^{2/3}}{h} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h}} = \pm\infty.$$

   This is different from the $\sqrt[3]{x}$ example because the limit is $\pm\infty$ rather than just $+\infty$.

4. **Discontinuity**

   Any function that is not continuous at a point cannot be differentiable at that point. In particular, if f is differentiable at a, then

   $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

   converges. But the bottom goes to zero, so the top must also go to zero, and we have

   $$\lim_{x \to a} f(x) = f(a),$$

   which is precisely what it means to be continuous.

   Conceptually, if the function isn't continuous, it isn't changing smoothly and so doesn't have a "speed" of change. Graphically, a function that has a disconnect in it doesn't have a clear tangent line.

   An example here is the Heaviside function $H(x)$. We have

   $$\lim_{h \to 0^+} \frac{H(h) - H(0)}{h} = \lim_{h \to 0^+} \frac{0}{h} = 0$$

but

$$\lim_{h\to 0^-} \frac{H(h) - H(0)}{h} = \lim_{h\to 0^-} \frac{-1}{h} = +\infty.$$

Since the one-sided limits aren't equal, the limit does not exist.

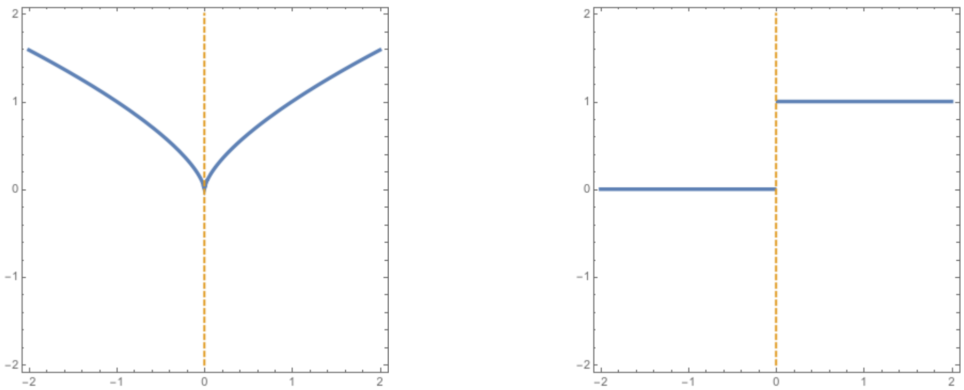


Figure 2.3: A cusp and a discontinuous function

The Derivative as a Function When we defined the derivative, we said it was a *number*: in definition 2.3 we defined the derivative *at a* to be the slope of the linear approximation *at a*. But this means we have a relationship where at each number a, we get a number $f'(a)$, and that means we have a function! Thus f' is a function and we can study it the way we did earlier functions.

Definition 2.8. The *derivative of a function f* is the function that takes in an input x and outputs

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Remark 2.9. On the first day of class, we said that functions don't have to take in and output numbers, it's just that the ones we're studying involve numbers. But if we want to get very abstract, "the derivative" is a function $\frac{d}{dx}$ which takes in a function f and gives back a new function f'. This is why we sometimes write $\frac{d}{dx}f(x) = f'(x)$.

Example 2.10.

1. If $f(x) = x^2 + 1$, we computed that $f'(x) = 2x$. The domain of f is all reals, and so is the domain of $f'(x)$.

2. If $g(x) = \sqrt{x}$ then $g'(x) = \frac{1}{2\sqrt{x}}$. The domain of g is all reals ≥ 0, and the domain of g' is all reals > 0.

3. We saw above that if $a(x) = |x|$, then the function $a'(x)$ is given by the piecewise formula

$$a'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{undefined} & x = 0 \end{cases} = \frac{|x|}{x}.$$

   The domain of a is all reals and the domain of a' is all reals except 0.

Given a function, we can compute the derivative. Since f' is a function we can ask about the derivative of the function f' at a point a.

Definition 2.11. Let f be a function which is differentiable at and near a point a. The *second derivative of f at a* is the derivative of the function $f'(x)$ at a, which is

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(x)}{h} = \frac{d^2 f}{dx^2}(a).$$

This is again a limit and may or may not exist.

Remark 2.12. The Leibniz notation for a second derivative is $\frac{d^2 f}{dx^2}$ and not $\frac{df^2}{dx^2}$. Conceptually, you can think of $\frac{d}{dx}$ as a function whose input is the function f and whose output is the derivative function f'. The second derivative results from applying this function twice, and thus is $\left(\frac{d}{dx}\right)^2 = \frac{d^2}{dx^2}$.

Example 2.13. What is the second derivative of $f(x) = x^3$ at $a = 2$?

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2 h + 3h^2 + h^3}{h} = \lim_{h \to 0} 3x^2 + 3h + h^2 = 3x^2.$$

$$\begin{aligned} f''(2) &= \lim_{h \to 0} \frac{f'(2+h) - f'(2)}{h} = \lim_{h \to 0} \frac{3(2+h)^2 - 3 \cdot 2^2}{h} = \lim_{h \to 0} \frac{3(4 + 4h + h^2) - 12}{h} \\ &= \lim_{h \to 0} \frac{12h + 3h^2}{h} = \lim_{h \to 0} 12 + 3h = 12. \end{aligned}$$

We won't say much more about the second derivative now, but we'll discuss it extensively in section 3.

2.3 Computing Derivatives

By now we're getting pretty tired of computing those examples over and over. In this section we'll come up with some techniques to make computation of derivatives easier.

1. **Constants** If c is a constant and $f(x) = c$ then $f'(x) = 0$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

Conceptually, a constant function never changes, so the rate of change is 0.

Geometrically, a constant function is a horizontal line; thus we think of the slope everywhere as being 0.

Example 2.14. $(3^{3^{3^3}})' = 0$.

2. If $f(x) = x$, then $f'(x) = 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - x}{h} = \lim_{h\to 0} 1 = 1.$$

Conceptually, if we have the "identity" function, then whenever we change the input then the output should change by exactly the same amount. Thus the rate of change is 1.

Geometrically, this is a line with slope 1.

3. If c is a constant and g is a function and $f(x) = c \cdot g(x)$, then $f'(x) = c(g'(x))$.

$$f'(x) = \lim_{h\to 0} \frac{cg(x+h) - cg(x)}{h} = c \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = c \cdot g'(x).$$

Conceptually, if changing x by a bit changes $g(x)$ by a certain amount, then it will change $2g(x)$ by twice that amount–multiplying by a scalar should just change the rate of change by the same amount everywhere.

Geometrically, multiplying by a constant is just stretching vertically–and all the slopes will be stretched by that same amount.

Example 2.15. If $f(x) = 5x$ then $f'(x) = (5 \cdot x)' = 5 \cdot x' = 5$.

4. If f and g are functions then $(f + g)'(x) = f'(x) + g'(x)$.

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x).\end{aligned}$$

Conceptually, if changing the input by a bit changes f by a certain amount and g by a different amount, then it changes $f + g$ by the sum of those two amounts–figure out how much it changes each part and then add them together to find out how much it changes the whole.

Geometrically, if we add two functions together it's just like stacking them on top of one another, so the slope at any point will be the sum of the slopes.

Example 2.16. Let $f(x) = 3x - 7$. Then $f'(x) = (3x)' - 7' = 3(x') - 0 = 3$.

This rule is really important but so far we can't do much with it–we don't have quite enough rules yet.

5. (Power Rule) If $f(x) = x^n$ where n is a positive integer, then $f'(x) = nx^{n-1}$. In fact, if $g(x) = x^r$ and r is any real number, then $g'(x) = rx^{r-1}$. We'll only prove this for integers, using the difference-of-nth-powers rule.

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{z^n - x^n}{z - x} = \lim_{z \to x} \frac{(z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})}{z - x} \\ &= \lim_{z \to x} z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1} = x^{n-1} + \cdots + x^{n-1} = nx^{n-1}. \end{aligned}$$

Now that we have this, we can compute all sorts of derivatives.

Example 2.17.
- $(x^2 + 1)' = 2x + 0 = 2x$.
- $(\sqrt{x})' = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.
- $(\sqrt[3]{x})' = (x^{1/3})' = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$.
- $(3\sqrt{x} + x^5 - 7)' = \frac{3}{2\sqrt{x}} + 5x^4 + 0$.

6. (Product Rule) If f and g are functions then $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Conceptually, we sort of know this already; if we add a bit on to f and a bit on to g, then we get $(f + f_h)(g + g_h) = fg + fg_h + gf_h + g_hf_h$, and in the limit we can treat g_hf_h as being zero. So this is the same as multiplying the bit we add to g with f, and multiplying the bit we add to f with g, and then adding the two.

Example 2.18. $((3x - 2)(x - 1))' = (3x^2 - 5x + 2)' = 6x - 5$.

Alternatively, $((3x-2)(x-1)' = (3x-2)'(x-1)+(3x-2)(x-1)' = 3\cdot(x-1)+1\cdot(3x-2) = 6x - 5$.

This rule isn't terribly important as long as we're only working with rational functions. Once we include anything else, like trig functions, it is critical.

Remark 2.19. We can get the power rule from the product rule instead of trying to get it directly.

7. (Quotient Rule): If f and g are functions then

$$(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

$$\begin{aligned}
(f/g)'(x) &= \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left(\lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h} + \lim_{h\to 0} \frac{f(x)g(x) - f(x)g(x+h)}{h} \right) \\
&= \frac{1}{g(x)^2} \left(g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \right) \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\end{aligned}$$

Example 2.20.

- $\left(\frac{x-1}{x^3}\right)' = (x^{-2} - x^{-3})' = -2x^{-3} + 3x^{-4}$.

  Alternatively,

  $$\left(\frac{x-1}{x^3}\right)' = \frac{(x-1)'x^3 - (x-1)3x^2}{x^6} = \frac{x^3 - 3x^3 + 3x^2}{x^6} = -2x^{-3} + 3x^{-4}.$$

- $\left(\frac{2+3x}{3-5x}\right)' = \frac{(2+3x)'(3-5x) - (2+3x)(3-5x)'}{(3-5x)^2} = \frac{9 - 15x + 10 + 15x}{(3-5x)^2} = \frac{19}{(3-5x)^2}$

2.4 Trigonometric derivatives

We cannot neglect the trigonometric functions—no matter how much we might wish to on occasion. All of the rules for trigonometric derivatives rely on what are known as the *angle addition formulas*:

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \qquad \cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b).$$

Note: you probably won't ever need to know these formulas again in this class. But I will need them for another page or so of these notes.

Using this we can compute

1.
$$\begin{aligned}(\sin(x))' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}\\
&= \left(\lim_{h\to 0}\frac{\sin(h)\cos(x)}{h}\right)+\left(\lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h}\right)\\
&= \cos(x)\lim_{h\to 0}\frac{\sin h}{h}+\sin(x)\lim_{h\to 0}\frac{\cos h-1}{h}\\
&= \cos(x)+\sin(x)\lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(h)+1)}\\
&= \cos(x)-\sin(x)\lim_{h\to 0}\frac{\sin^2(h)}{h(\cos(h)+1)}\\
&= \cos(x)-\sin(x)\left(\lim_{h\to 0}\frac{\sin(h)}{\cos(h)+1}\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\\
&= \cos(x)-\sin(x)\cdot 0\cdot 1 = \cos(x).\end{aligned}$$

2. A similar argument shows that $(\cos(x))' = -\sin(x)$.

Further using the product and quotient rules, we observe that

- $$(\tan(x))' = \left(\frac{\sin x}{\cos x}\right)' \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$
- $$(\cot(x))' = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2(x)-\cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$
- $$(\sec(x))' = \left(\frac{1}{\cos x}\right)' = \frac{0+\sin x}{\cos^2(x)} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \sec(x)\tan(x)$$
- $$(\csc(x))' = \left(\frac{1}{\sin x}\right)' = \frac{0-\cos(x)}{\sin^2(x)} = \frac{-\cos x}{\sin x}\cdot\frac{1}{\sin x} = -\csc(x)\cot(x).$$

Remember that as long as you know the derivatives of sin and cos you can always compute these four derivatives whenever you need them.

Example 2.21. 1. If $f(t) = 3\sin t + \cos t$, then $f'(t) = 3\cos t - \sin t$.

2. Find the tangent line to $y = 6\cos x$ at $(\pi/3, 3)$.

   We see that $y' = -6\sin x$, and thus when $x = \pi/3$ we have $y' = -3\sqrt{3}$. Recalling that the equation of our line is $y = m(x - x_0) + f(x_0)$, we have the equation $y = -3\sqrt{3}(x - \pi/3) + 3$.

3. IF $g(\theta) = \theta \sin\theta + \frac{\cos\theta}{\theta}$, then
$$g'(\theta) = (\sin\theta + \theta\cos\theta) + \frac{-\theta\sin\theta - \cos\theta}{\theta^2}.$$

4. If $h(x) = \frac{x}{2-\tan x}$, then
$$h'(x) = \frac{(2-\tan x) + x\sec^2 x}{(2-\tan x)^2}.$$

5. We can also compute second derivatives. $\sin'' x = -\sin x$. $\cos'' x = -\cos x$.
$$\tan'' x = (\sec x \sec x)' = \sec x \tan x \sec x + \sec x \tan x \sec x = 2\sec^2 x \tan x.$$

2.5 The Chain Rule

To start with an example, suppose $g(x) = (\sin x)^2$. Then
$$g'(x) = ((\sin x)(\sin x))' = \cos x \sin x + \cos x \sin x = 2\sin x \cos x.$$

Remembering that $(x^2)' = 2x$, we notice that this looks suggestive. It also leads us to ask what happens when we build up functions by composition, that is, plugging one function into another, as we have here.

If we want to freely build complex functions from simple ones, we need to be able to combine them in chains. Remember that we define the function $f \circ g$ by $(f \circ g)(x) = f(g(x))$; we take our input x, plug it into g, and then take the output $g(x)$ and plug it into f.

We can see how this is useful in two different ways. First, as we saw earlier, it lets us build up functions.

1. $(x+1)^2 = (f \circ g)(x)$ where $g(x) = x+1$ and $f(x) = x^2$.
2. $(x^2+1)^2 = (f \circ g)(x)$ where $g(x) = x^2+1$ and $f(x) = x^2$.
3. $\sin^2(x) = (f \circ g)(x)$ where $g(x) = \sin x$ and $f(x) = x^2$.

Second, sometimes composition of functions really is the best way to describe what's going on, especially when you have a "causal chain" where one process causes a second which causes a third. For instance, suppose you're driving up a mountain at 2 km/hr, and the temperature drops 6.5° C per kilometer of altitude. You can think about your temperature as a function of your height, which is itself a function of the time; then the numbers I gave you are the rates of change, or derivatives, of each function.

It's not that hard to convince yourself that you'll get colder by about 13° C per hour. Does this work in general?

Proposition 2.22 (Chain Rule)**.** *Suppose f and g are functions, such that g is differentiable at a and f is differentiable at $g(a)$. Then $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.*

Proof.

$$\begin{aligned}(f \circ g)'(a) &= \lim_{h \to 0} \frac{(f \circ g)(a+h) - (f \circ g)(a)}{h} \\ &= \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \cdot \frac{g(a+h) - g(a)}{h} \\ &= \left(\lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)}\right)\left(\lim_{h \to 0} \frac{g(a+h) - g(a)}{h}\right) \\ &= f'(g(a)) \cdot g'(a).\end{aligned}$$

□

Remark 2.23.

1. When we write $f'(g(x))$, we mean the function f' evaluated at the point $g(x)$, or in other words, the derivative of f at the point $g(x)$.
2. It can be helpful as a way of remembering the chain rule that
$$\frac{d(f \circ g)}{dx} = \frac{d(f \circ g)}{dg} \cdot \frac{dg}{x}.$$
Don't take this too seriously as actively meaning anything, since it only sort of does, but it's quite helpful for the memory.

Example 2.24.

1. $(x+1)^2 = (f \circ g)(x)$ where $g(x) = x + 1$ and $f(x) = x^2$. Then $f'(x) = 2x$ and $g'(x) = 1$, so
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2(g(x)) \cdot 1 = 2(x+1) \cdot 1 = 2x + 2.$$
Sanity check:
$$(f \circ g)'(x) = (x^2 + 2x + 1)' = 2x + 2.$$
2. $(x^2+1)^2 = (f \circ g)(x)$ where $g(x) = x^2 + 1$ and $f(x) = x^2$. Then $f' = 2x$, $g' = 2x$, and
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 2(g(x)) \cdot 2x = 2(x^2+1) \cdot 2x = 4x^3 + 4x.$$
Sanity check:
$$(f \circ g)'(x) = (x^4 + 2x^2 + 1)' = 4x^3 + 4x.$$
3. $\sin^2(x) = (f \circ g)(x)$ where $g(x) = \sin x$ and $f(x) = x^2$. Then $f'(x) = 2x$, $g'(x) = \cos x$, and we have
$$(f \circ g)'(x) = 2(g(x)) \cdot \cos x = 2(\sin x) \cos x.$$

4. $\cos(3x) = (f \circ g)(x)$ where $f(x) = \cos(x)$ and $g(x) = 3x$. Then $f'(x) = -\sin(x)$ and $g'(x) = 3$ and
$$(f \circ g)'(x) = -\sin(3x) \cdot 3.$$

5. $\sin(x^2) = (f \circ g)(x)$ where $f(x) = \sin(x)$ and $g(x) = x^2$. Then $f'(x) = \cos x, g'(x) = 2x$, and
$$(f \circ g)'(x) = \cos(g(x)) \cdot 2x = 2x\cos(x^2).$$

6. If $f(x)$ is any function, then we can write $(f(x))^r$ as $(g \circ f)(x)$ where $g(x) = x^r$. Then
$$\frac{d}{dx}(f(x)^r = (g \circ f)'(x) = r(f(x))^{r-1} \cdot f'(x).$$

7. The derivative of $\sec(5x)$ is $\sec(5x)\tan(5x)5$.

8. What is the derivative of $\frac{1}{\sqrt[3]{x^4-12x+1}}$? We can view this as $(x^4 - 12x + 1)^{-1/3}$, and using the chain rule, we have
$$\frac{d}{dx}\frac{1}{\sqrt[3]{x^4 - 12x + 1}} = \frac{-1}{3}(x^4 - 12x + 1)^{-4/3} \cdot (4x^3 - 12).$$

9. What is the derivative of $\sec^2(x)$? By the chain rule this is $2 \cdot \sec(x) \cdot \sec'(x) = 2\sec(x) \cdot \sec(x)\tan(x) = 2\sec^2(x)\tan(x)$.

10. What is the derivative of $\sec^4(x)$? We get $4\sec^3(x)\sec'(x) = 4\sec^3(x)\sec(x)\tan(x) = 4\sec^4(x)\tan(x)$.

11. Sometimes we have to nest the chain rule. What is the derivative of $\sqrt{x^3 + \sqrt{x^2+1}}$? We can pull this apart slowly.
$$\begin{aligned}\frac{d}{dx}\sqrt{x^3 + \sqrt{x^2+1}} &= \frac{1}{2}(x^3 + \sqrt{x^2+1})^{-1/2} \cdot \left(\frac{d}{dx}\left(x^3 + \sqrt{x^2+1}\right)\right)\\ &= \frac{1}{2\sqrt{x^3 + \sqrt{x^2+1}}}\left(3x^2 + \frac{1}{2}(x^2+1)^{-1/2} \cdot \left(\frac{d}{dx}x^2 + 1\right)\right)\\ &= \frac{3x^2 + \frac{2x}{2\sqrt{x^2+1}}}{2\sqrt{x^3 + \sqrt{x^2+1}}}\end{aligned}$$

As we have just seen the chain rule can stack, or chain together. As functions get more complicated we will have to use multiple applications of the product rule, quotient rule, and chain rule to pull our derivative apart.

Example 2.25. Find
$$\frac{d}{dx}\sec(x^2+\sqrt{x^3+1}).$$

$$\begin{aligned}\frac{d}{dx}\sec(x^2+\sqrt{x^3+1}) &= \sec(x^2+\sqrt{x^3+1})\cdot\tan(x^2+\sqrt{x^3+1})\cdot(x^2+\sqrt{x^3+1})' \\ &= \sec(x^2+\sqrt{x^3+1})\cdot\tan(x^2+\sqrt{x^3+1})\cdot(2x+\frac{1}{2}(x^3+1)^{-1/2}\cdot 3x^2)\end{aligned}$$

Example 2.26 (recitation)**.** Find
$$\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1}$$

$$\begin{aligned}\frac{d}{dx}\frac{\sin(x^2)+\sin^2(x)}{x^2+1} &= \frac{(\sin(x^2)+\sin^2(x))'(x^2+1)-2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2} \\ &= \frac{(\cos(x^2)\cdot 2x+2\sin(x)\cos(x))(x^2+1)-2x(\sin(x^2)+\sin^2(x))}{(x^2+1)^2}.\end{aligned}$$

We can keep going with increasingly complicated problems, basically until we get bored. These are really good practice for making sure you understand how the rules fit together.

Example 2.27. Find
$$\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}}$$

$$\begin{aligned}\frac{d}{dx}\sqrt{\frac{\sqrt{x}+1}{(\cos x+1)^2}} &= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2}\cdot\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)' \\ &= \frac{1}{2}\left(\frac{\sqrt{x}+1}{(\cos x+1)^2}\right)^{-1/2}\cdot\frac{\frac{1}{2}x^{-1/2}(\cos x+1)^2-2(\cos x+1)(-\sin x)(\sqrt{x}+1)}{(\cos x+1)^4}\end{aligned}$$

Example 2.28 (Bonus)**.** Calculate
$$\frac{d}{dx}\left(\frac{\sin^2\left(\frac{x^2+1}{\sqrt{x-1}}\right)+\sqrt{x^3-2}}{\cos(\sqrt{x^2+1}+1)-\tan(x^4+3)}\right)^{5/3}$$

2.6 Linear Approximation

In section 2.1 we defined the derivative in terms of approximation. We took an *algebraic* approach where we wanted to approximate a function with a line, and found a number $f'(a)$ that made the line $y=f'(a)(x-a)+f(a)$ approximate the function f as well as possible.

In this section we want to return to this idea, now that we know how to compute derivatives. Then in section 2.7 we'll see how we can use this to model physical, economic, and other practical phenomena. Finally in section 2.8 we'll take a *geometric* perspective, where we see how we can use derivatives to understand geometric pictures and graphs of functions.

We know that if we have a function $f(x)$ and know what it looks like at a point a, we can use the derivative to give a linear approximation

$$f(x) \approx f(a) + f'(a)(x-a).$$

Example 2.29. We can find an estimate of 2.1^5.

To a "zeroth approximation", we might say that $2.1^5 \approx 2^5 = 32$; that's the approach we took in section 4. We can now use the derivative to refine that estimate. We take $f(x) = x^5$ and $a = 2$. Then $f'(x) = 5x^4$, so we have $f(2) = 32, f'(2) = 80$, and

$$f(2.1) \approx 80(2.1-2) + 32 = 40.$$

The exact answer is 40.841, so this estimate is pretty good!

What if we approximate $(2.5)^5$ using $a = 2$. What if we pproximate 3^5? We have

$$\begin{aligned} (2.5)^5 &\approx 80 \cdot (2.5-2) + 32 = 72 \\ 3^5 &\approx 80 \cdot (3-2) + 32 = 112. \end{aligned}$$

The true answers are 97.6563 and 243. These estimates are not especially good. This is because 3 is actually not very close to 2—especially proportionately. Of course, it's not that hard to compute 3^5 directly.

These methods are best when $x - a$ is very small relative to everything else. We often use them in the real world for $x - a < .1$ or so.

Example 2.30. Let's approximate $\sqrt[3]{28}$ and $\sqrt[4]{82}$.

We take $a = 27$ and $a = 81$ respectively.

$$\begin{aligned} \sqrt[3]{28} &\approx \frac{1}{3}(27)^{-2/3}(28-27) + 3 = \frac{1}{27} + 3 \approx 3.03704 \\ \sqrt[4]{82} &\approx \frac{1}{4}(81)^{-3/4}(82-81) + 3 = \frac{1}{108} + 3 \approx 3.00926. \end{aligned}$$

The true answers are approximately 3.03659 and 3.00922 respectively.

Now we'll approximate 28^3 and 82^4 using the same base points

We have

$$\begin{aligned} 28^3 &\approx 3(27)^2(28-27) + 27^3 = 21870 \\ 82^4 &\approx 4(81)^3(82-81) + 81^4 = 45172485 \end{aligned}$$

In contrast the true answers are 21952 and 45172485.

These approximations aren't *terrible* but they aren't very good either. Since the derivative is changing quickly here (the second derivatives are $6 \cdot 27$ and $12 \cdot 81^2$ respectively), the approximation won't be very good.

Example 2.31. If you take $a = 0$ and $f(x) = x^{10}$, we can use a linear approximation to approximate $f(2)$. We have $f'(x) = 10x^9$, so we have $f'(0) = 0$, and thus

$$f(2) \approx 0(2 - 0) + 0 = 0.$$

Since the true answer is 1024, this is not very good. What if we use $a = 1$ instead? If we take $a = 1$, we have

$$f(2) \approx 10(2 - 1) + 1 = 11.$$

This is a little better, but still not good. In essence, the derivative is changing so quickly that the tangent line approximation is not very good over those distances. Later, in section 4.1, we'll talk a little bit about how we can handle this situation better.

There are a few specific linear approximation *formulas* that come up really frequently in other applications, enough to get their own names. I want to take a moment to look at each of them.

Example 2.32 (Binomial Approximation)**.** As a warmup, let's approximate $(1.01)^{10}$. Our function is $f(x) = x^{10}$ and our $a = 1$. So $f(a) = 1$ and $f'(a) = 10a^9 = 10$. Then we have

$$f(1.01) \approx 10(1.01 - 1) + 1 = 1.1.$$

The true answer is about 1.10462.

Now let's approximate $(1.01)^\alpha$ where $\alpha \neq 0$ is some constant. (The letter α is a Greek lower-case "a". I'm using it here instead of the friendlier n because it's fairly standard for the formula we're developing.)

We have $f(x) = x^\alpha$, so $f'(x) = \alpha x^{\alpha-1}$. We again have $f(1) = 1$ and $f'(1) = \alpha(1)^{\alpha-1} = \alpha$, so

$$f(1.01) \approx \alpha(1.01 - 1) + 1 = 1 + \alpha/100.$$

Now let's get the fully general useful formula: approximate $(1 + x)^\alpha$ where x is some small number and $\alpha \neq 0$ is a constant. (This rule is the called the "binomial approximation" and is often useful in physics and engineering).

We still take $f(x) = x^\alpha$ and $a = 1$. But we compute

$$f(1 + x) \approx 1 + \alpha(1 + x - 1) = 1 + \alpha x.$$

It is probably more helpful in the long run to think about $f(x) = (1+x)^\alpha$, though. Then we have $f'(x) = \alpha(1 + x)^{\alpha-1}$, and we get

$$f(x) \approx 1 + \alpha x.$$

Example 2.33 (Small Angle Approximation)**.** Let's find a formula to approximate $\sin(x)$ when x is small. You might think of this as the revenge of the Small Angle Approximation from section 1.5.

We take $a = 0$. Then since $\sin'(x) = \cos(x)$ and so $\sin'(0) = \cos(0) = 1$, we have

$$\sin(x) \approx 1(x - 0) + 0 = x.$$

Thus for small angles, $\sin(x)$ is approximately just x! For instance, our formula says that $\sin(.05) \approx .05$, where the true answer is about .04998. So this is pretty good. In fact, we compute that $\sin''(0) = -\sin(0) = 0$. Since the second derivative is zero, we expect the linear approximation to work well.

That means that in a lot of calculations, if we have a formula with a lot of sines in it, as long as our angles are small we can replace every $\sin(x)$ with an x without losing too much. And that's much easier to think about.

We can do the same thing for cosine. We compute that $\cos'(x) = -\sin(x)$ so $\cos'(0) = 0$. Then

$$\cos(x) \approx 0(x - 0) + 1 = 1.$$

This is actually a constant! The line that fits $\cos(x)$ best near 0 is just the horizontal line $y = 1$.

We can calculate, e.g., that $\cos(.05) \approx 1$, where the true answer is about .9986. This is also pretty good, but the approximation isn't quite as good as the one for sine. We compute that $\cos''(0) = -\cos(0) = -1$; while the second derivative isn't huge, it isn't trivial either.

Example 2.34 (Geometric Series)**.** Let's find a formula to linearly approximate $f(x) = \dfrac{1}{1-x}$ near $x = 0$.

We compute that $f'(x) = (1 - x)^{-2} = \frac{1}{(1-x)^2}$. Then

$$f(x) \approx 1 + x.$$

This is a special case of what's known as the geometric series formula.

You might ask why we did the slightly funky $\frac{1}{1-x}$ instead of the more normal $\frac{1}{x}$. After thinking about it for a bit, you'll notice that wee can't approximate $\frac{1}{x}$ near zero at all! We see that f is undefined at 0, and equally importantly, $f'(x) = -1/x^2$ is also undefined at zero. So there's no linear approximation.

But if we want to,, we can linearly approximate $f(x) = 1/x$ near 1. We have $f(1) = 1$ and $f'(1) = -(1)^{-2} = -1$ so

$$f(x) \approx 1 - (x-1) = 2 - x.$$

Finally, a bonus fun fact to notice.

Example 2.35. Let's find a formula to approximate $f(x) = x^3 + 3x^2 + 5x + 1$ near $a = 0$. What do you notice? Why does that happen?

We have $f(0) = 1$ and $f'(x) = 3x^2 + 6x + 5$ so $f'(0) = 5$. Thus

$$f(x) \approx 1 + 5x.$$

This is exactly what you get if you take the original polynomial and cut off all the terms of degree higher than 1.

This makes sense, because we're looking for the closest we can get to f without using terms of degree higher than 1.

2.7 Speed and Rates of Change

In this section we'll develop a second way of thinking about the derivative. We'll ask a different question, and see that the derivative is also an answer to that question. We'll talk a little bit about why the two different questions are secretly the same, and thus explain why you might *care* about linear approximation, even if you aren't as much of a nerd for algebra as I am.

2.7.1 The Problem of Speed

An important concept in physics is *speed*, which is defined to be distance covered divided by time spent. That is, $v = \frac{\Delta x}{\Delta t}$. In particular, if your position at time t is given by the function $p(t)$, then your average speed between time t_0 and time t_1 is

$$v = \frac{p(t_1) - p(t_0)}{t_1 - t_0}.$$

This formula should look familiar. It is the slope of a line through the points $(t_0, p(t_0))$ and $(t_1, p(t_1))$. It is *not* the derivative of p, because we didn't take a limit. It is instead a "difference quotient", which is really a fancy way of saying the slope of a line.

Example 2.36. For example, on Earth dropped objects fall about $p(t) = 5t^2$ meters after t seconds. The average speed between time $t = 1$ and time $t = 2$ is

$$v = \frac{p(2) - p(1)}{2 - 1} = \frac{20 - 5}{1} = 15\text{m/s}$$

and the average speed between time $t = 3$ and time $t = 1$ is

$$v = \frac{p(3) - f(1)}{3 - 1} = \frac{45 - 5}{3 - 1} = 20\text{m/s}.$$

It's useful here to look at the units. We know that the result is a speed, so comes out in m/s. But how do we know we get those units? We have to think a bit about what the function p is actually doing.

The function p gives us position as a function of time. Thus the *inputs* to p are given in seconds, and the *outputs* are given in meters. So it's not really fully correct to say that $p(t) = 5t^2$; that would suggest that $p(1\text{s}) = 5(1\text{s})^2 = 5\text{s}^2$. But your position isn't described in square seconds!

Instead, we would write something like $p(t\text{seconds}) = 5t^2\text{m}$. The function takes in seconds as inputs, and gives meters as outputs. Thus our last calculation properly should have been

$$v = \frac{p(3\text{s}) - f(1\text{s})}{3\text{s} - 1\text{s}} = \frac{45\text{m} - 5\text{m}}{3\text{s} - 1\text{s}} = 20\text{m/s}.$$

We see that the numerator—which is made up of the outputs of p—has units of meters, while the denominator, which is made up of the inputs of p, has units of seconds. So the entire fraction has units of m/s, which is what it should be.

We can give a more general formula. What's the average speed between time $t_0 = 1$ and time $t_1 = t$? We have

$$v = \frac{p(t\text{s}) - p(1\text{s})}{t\text{s} - 1\text{s}} = \frac{5t^2\text{m} - 5\text{m}}{t\text{s} - 1\text{s}} = 5(t + 1)\frac{t - 1}{t - 1}\text{m/s}.$$

As long as $t \neq 1$, this gives us a formula for average speed between time t and time 1: the average speed is $5(t + 1)$m/s. But what if we want to know the speed "at" the time $t = 1$?

On some level, this question doesn't make any sense. Speed is defined as the change in distance divided by the change in time; if time doesn't change, and distance doesn't change, then this doesn't really mean anything. Maybe what we really mean is, what's a good estimate of our average speed, as long as our time is close to $t = 1$? Our average speed depends on the exact interval we choose; the speed from $t = 1$ to $t = 2$ isn't the same as the speed from $t = 1$ to $t = 1.1$. But can we find one number that gives a good estimate?

This should make you think of the limit idea from section 1.3. We can find a good estimate of the speed from time 1 to time t by taking a limit as t approaches 1. Thus we define your *instantaneous speed* or *speed at time* t_0 to be

$$\lim_{t_1 \to t_0} \frac{p(t_1) - p(t_0)}{t_1 - t_0} = \lim_{h \to t_0} \frac{p(t_0 + h) - p(t_0)}{h}.$$

Notice that since the function p has input in seconds and output in meters, the instantaneous speed will be in m/s, as it should be. But also notice that this formula is just the definition of the derivative of p.

Thus from the previous example, we can see that the instantaneous speed at time $t_0 = 1$ is

$$v(1\text{s}) = p'(1\text{s}) = \lim_{t \to 1} 5(t+1)\frac{t-1}{t-1}\text{m/s} = 10\text{m/s}.$$

Alternatively, we know that $p(t) = 5t^2$, so by our derivative rules we know that $p'(t) = 10t$ and thus $p'(1) = 10$. Once we add units, we have $p'(t\text{s}) = 10t\text{m/s}$ and thus $p'(1\text{s}) = 10\text{m/s}$.

The derivative of a function has different units from the original function. Since the derivative is given by a formula with output in the numerator and input in the denominator, the derivative will have the units of the output per units of input.

We can take this one step further and look at the derivative of p'. The function p' takes in a time and outputs a speed; its derivative will be

$$p''(t_0\text{s}) = \lim_{t \to t_0} \frac{p'(t\text{s}) - p'(t_0\text{s})}{t\text{s} - t_0\text{s}}.$$

The units of the denominator are still seconds; but the units of the top are m/s, so the second derivative takes in seconds and outputs meters per second *per second*, or m/s^2. This makes sense: the second derivative is the change in the first derivative, so p'' tells us how quickly the speed is changing. So it tells us how many meters per second your speed changes each second. This is otherwise known as "acceleration".

Once we have the speed of a particle in terms of its derivative, we can apply it to do the sort of things we've already been doing. So for instance, we can ask how far a dropped object will have fallen after 2.2 seconds. We could calculate this exactly, but we can also approximate:

$$p(2.2\text{s}) \approx p(2\text{s}) + p'(2\text{s})(2.2\text{s} - 2\text{s}) = 20\text{m} + 10\text{m/s}(.2\text{s}) = 22\text{m}.$$

How does all this relate to linear approximation? We know that speed is change in distance over time. Another way of saying that is that our final position is our initial position, plus speed times time.

$$p(t) = p(0) + v_{\text{average}}(t - 0).$$

If our speed varies over time, this isn't terribly helpful: we can only compute average speed by knowing our initial and final position. If we only know our speed "at" each moment, this doesn't work—and making it work precisely involves *integrals*, which we will develop in sections 5 and 6.

But if the length of time is small, we can make a pretty good guess by assuming our speed is constant. Thus we compute our instantaneous speed at time 0, and wee have the approximate formula

$$p(t) \approx p(0) + v_0(t - 0).$$

And this is precisely the linear approximation formula we started with in 2.1.

Remark 2.37. This is basically how we reason about speed in real life. If you're driving fifteen miles and your friend calls you and asks how long you'll take, you might say "Well, traffic isn't too bad; I'm going about 30 miles per hour. So I should be there in about half an hour". This doesn't mean you'll get there in exactly half an hour. Traffic might get better or worse, and you might speed up or slow down. But your best guess of your average speed is your speed right now.

Of course, that's not always your best guess. If you're driving into the city you might know that you're about to hit bad traffic. Or if you can see the end of your traffic jam, you might know you're about to speed up. In either case, this is like having information about the second derivative, and you can refine your guess.

The worst-case version of this thought process is the old Windows download boxes, which would give an estimate of how long a file transfer would take. But this estimate was a simple linear approximation of remaining file size divided by your current download speed—and download speeds would vary wildly from second to second. So you'd see an estimate jump from thirty minutes to two hours to five minutes and back up to forty minutes, all within the space of thirty seconds.

2.7.2 Other Rates of Change

We used this to think about physical speed as we move from one location to another. But the same logic applies to basically any time we have a physical process with change over time. If you know how quickly the output is changing "right now", you can use that to build a linear model of what the output will look like over time. And that means that any rate of change is, fundamentally, a derivative.

Another way of thinking about the derivative is the difference between "stocks" and "flows". If your function measures the *level* or something, then the derivative measures the rate at which the level is changing. If the function measures the amount of something you have in stock, then the derivative measures the rate at which new stock is flowing in or out of your warehouse.

Example 2.38 (Debt and Deficit)**.** A lot of discussions of economics and public policy address the deficit and the debt. The "deficit" and the "debt" are easy to confuse but importantly different, in a way that maps cleanly to the idea of a derivative.

A "deficit" is the amount of money that is currently owed; it is measured in dollars (or euro or yen or some other currency). The current US national deficit is approximately \$22 trillion.

A "deficit" is the rate at which the debt is increasing. So the national deficit is currently about \$1 trillion. This means we expect the debt next year to be about \$1 trillion bigger than the debt this year.

Mathematically we can define a function $D(t)$ which takes in the year and outputs the number of dollars owed. Then the annual deficit is

$$\frac{D((t+1)\mathrm{y}) - D(t\mathrm{y})}{1\mathrm{y}}.$$

This isn't a derivative, since there's no limit; this is a *difference quotient* that measures a discrete change in debt over a discrete time. It's analogous to average speed, not instantaneous speed.

But we could imagine asking how the deficit is changing from month to month, or from week to week, or from hour to hour. We can take a limit as the time between $t + h$ and t goes to zero, and then the deficit would be the derivative of debt. The function $D'(t)$ will take in years, and output dollars per year.

What about the second derivative? The function D'' will take in years, and output the yearly change in the deficit, measured in dollars per year per year. When people talk about whether the deficit is going up or down, they are looking at the second derivative of the debt.

Example 2.39 (Inflation)**.** We can make a similar point about inflation, and make fun of Richard Nixon at the same time.

Roughly speaking, inflation is the change in the *price level*, which measures how the value of money changes over time. Thus inflation is a rate of change, and thus a derivative. If we oversimplify and measure the price level as the number of liters of gas you can buy with a dollar, then inflation is measured in liters per dollar per year.

In the seventies, inflation was a major political topic, because inflation was both high and rising. What does it mean to say inflation is rising? That's a *second derivative*. Inflation is the rate at which the price level is changing, but that rate is itself increasing.

In Nixon's reelection campaign, he couldn't say inflation was low, because it wasn't. And he couldn't even say it was falling, because it wasn't. So instead he said that "the rate at which the rate of inflation is increasing is decreasing". That's terrible sentence, even before we unpack it into "the rate at which the rate at which the price level is increasing is increasing is decreasing". (I promise that sentence wasn't me losing control of my keyboard.)

I've heard that this is the only known use of the third derivative in political messaging.

Both of these examples have one very important trait in common. The position function $p(t)$ and the debt function $D(t)$ output different types of things with different units, but they both take *time* as an input. But it's easy for a function to take inputs other than time, and these functions are often physically important and meaningful.

One common place they show up is in economics. Economics cares a lot about so called "marginal" effects.

Example 2.40 (Marginal Revenue)**.** If you're deciding how many machines to buy, what really matters isn't the total cost of the machines and the total revenue they'll make you. Instead, you need to ask how much more you'll have to spend to get *one* more machine, and how much more revenue that one machine will get you. (This is called "marginal thinking", because we care about the effect of getting one more machine on the margin.)

Any of these marginal effects are implicitly asking for a derivative. So suppose we have some revenue curve where $R(m) = 100m - m^2$: your total revenue is \$100 for every machine, minus upkeep costs of the square of the number of machines you have. So with one machine, you make \$99; with two machines, you make \$196; with ten machines you make \$900. The units of the input is "machines" and the units of the output are "dollars".

We compute $R'(m) = 100 - 2m$; each new machine adds roughly \$100 of revenue, minus 2 times the number of machines you already have. Thus the marginal revenue of the first machine is about \$98, and the marginal revenue of the tenth machine is about \$80. We can see that the fiftieth machine has a marginal revenue of \$0; this is our break-even point, where adding another machine neither helps nor hurts. The sixtieth machine has a marginal revenue of about $-\$20$, and we actually lose money by adding it! The units of this derivative are "dollars per machine"; how many more dollars will you get by adding a machine?

But of course the actual revenue of 50 machines is $R(50) = 5000 - 2500 = 2500$ dollars. The actual revenue of 60 machines is $R(60) = 6000 - 3600 = 2400$ dollars, which is less than $R(50)$ but still positive.

Example 2.41 (Marginal Cost)**.** We also often talk about marginal cost. Suppose the cost of buying m machines is $C(m) = 5000 + 10m + .05m^2$. There's some start-up cost to having any machines at all; then each machine costs a bit more than the previous one. The units of the input are "machines" and the units of output are "dollars".

We can see that $C(1) = 5010.05$, and $C(10) = 5105$. Even $C(100) = 6500$ is not that much bigger than $C(1)$.

The marginal cost would be $C'(m) = 10 + .1m$. We have to pay a huge sum to have any machines at all, but each new machine we add costs only 10 plus a tenth of the number of machines we have. So the cost of adding the hundredth machine is about $C'(100) = 20$, which checks out with the numbers we computed earlier. The units of the derivative are, again, dollars per machine.

This shows a really big separation between marginal and average cost. The total cost of all our machines is really high; if this cost is paired with the revenue from the previous example, we'll continually lose money no matter what we do. But once we've already eaten our sunk costs, the marginal cost of adding one more machine is pretty low, so we should go ahead and get a lot of them.

Example 2.42 (Ohm's Law)**.** In physics and electrical engineering, Ohm's Law tells us that current is equal to voltage over resistance, or $I = V/R$. (Here current is generally measured in amperes, voltage in volts, and resistance in, essentially, volts per amp).

The default assumption in most physics problems is that resistance is constant, a property of whatever material you're putting current through. So we have the function $I(V) = \frac{1}{R}V$, which is a linear function and simple to work with.

But this is just an approximation! Most materials will actually have their resistance change as the voltage applied to them changes, so the equation above is just a linear approximation to the actual relationship between current and voltage. This means that the slope $\frac{1}{R}$ is really a derivative.

An incandescent lightbulb works by running a current through a metal wire until it heats up. But as the heat of the wire increases, the resistance goes up. Thus the graph of current as a function of voltage is curving down; the higher the voltage, the less extra current you get from adding another volt. This means that the derivative $\frac{dI}{dV}$ is large when V is small, but small when V is large.

A diode is a material that does the opposite. Resistance is high when the voltage is low, but past some transition point the resistance drops and becomes very low. This means that the derivative is large when V is small, and then small when V is large. The graph of I as a function of V will curve up.

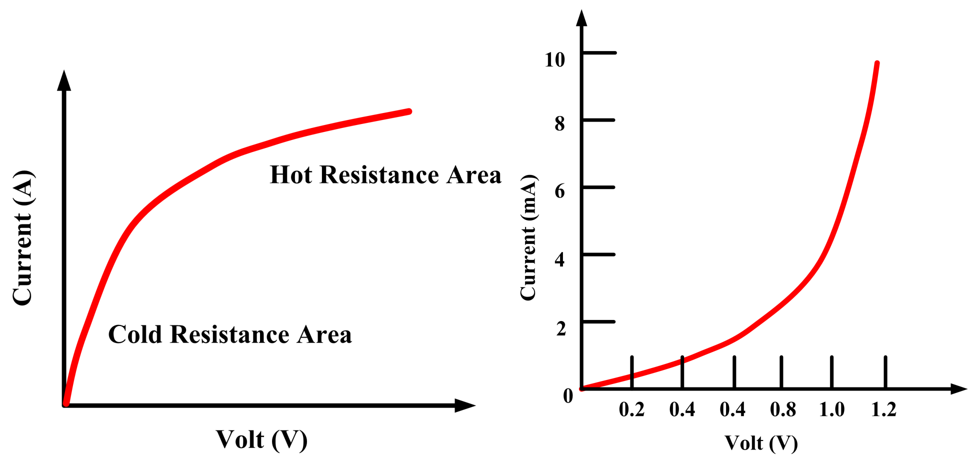


Figure 2.4: Current as a function of resistance for an incandescent bulb filament (left) and a diode (right)
Figures from Nonlinear Resistors — Characteristics Curves of Nonlinear Devices at `https://electricalacademia.com`

In practice. engineers mostly don't want to worry about the whole curve. If they know about what voltage their devices will experience, they don't need to worry what happens in other places. So they take the local linear approximation, call that "the resistance", and use the equation $I = I_0 + \frac{1}{R}(V - V_0)$. And this is just the linear approximation equation we've been using all class.

Example 2.43 (Price Elasticity of Demand)**.** Another common economics question is to see how the demand for a product relates to its price. We can define a function $Q(p)$ that takes in a price in dollars, and outputs the quantity of items that will be bought. So if $Q(p) = 10000 - 10p$, this means that if the price is \$100 then people will buy $Q(100) = 10000 - 1000 = 9000$ widgets.

What's the derivative here? The function $Q'(p)$ takes in a price in dollars and outputs a number of widgets per dollar. It tells you how the quantity demanded changes in response to changes in the price. Thus we see that since $Q'(p) = -10$, we expect to sell ten fewer widgets for each dollar we raise the price.

(Economists call this the Price Elasticity of Demand: "elasticity" is how quickly one thing responds to changes in another thing. So any time the term "elasticity" shows up in economics, there's a derivative inolved somewhere).

What if instead we had the function $Q(p) = 10000 - 5p^2$? Now we see that changing the price doesn't have a huge effect if the price is already small, but it has a dramatic effect if the price is big. We compute that $Q'(p) = -10p$. This means that increasing the price by one dollar will decrease the quantity demanded by ten widgets for every dollar of the price.

Thus if the current price is \$10, we expect raising the price to \$11 to reduce sales by about a hundred widgets. If the current price is \$30 then raising the price will lose us nine hundred widgets in sales.

2.8 Tangent Lines

In this section we'll introduce a third perspective on the derivative. We saw first an *algebraic* perspective, thinking about linear approximation, then a *physical* perspective thinking about rates of change. Now we'll take a *geometric* perspective.

Classically mathematicians were really interested in geometry, which was tied up deeply in questions of philosophy and theology. One obvious-to-them geometric question was to try to find a line *tangent* to the graph of some function.

Definition 2.44. A line that touches a curve at one point without crossing it is *tangent* to the curve at that point, and we call such a line a *tangent line* (from Latin *tangere* "to touch".)

A line crossing a curve in two points is called a *secant* line. (from Latin *secare* "to cut").

Just as the tangent of an angle is the length of a (specific) tangent line segment, the secant of an angle is the length of a (specific) secant line segment.

Suppose we want to find the tangent line to a graph at a point $(a, f(a))$. We need either two points, or a point and a slope. Clearly we have one point. The derivative gives a slope, but why is it the *right* slope?

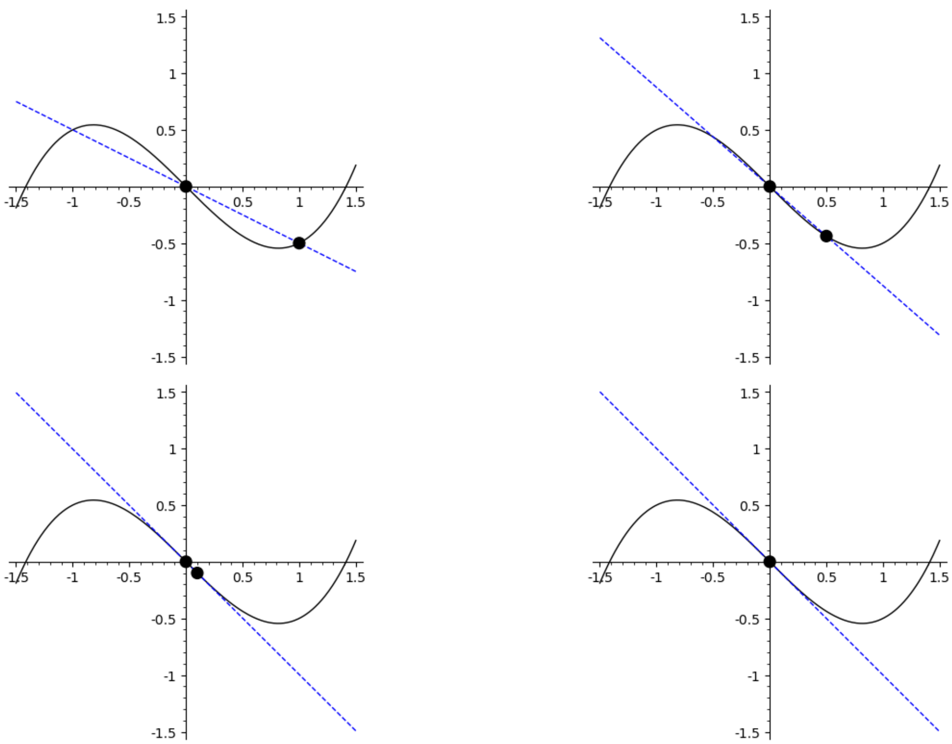
If we know another point $(b, f(b))$, then we can use the two-points formula to write the equation of a line through those two points:

$$f(x) - f(b) = \frac{f(b) - f(a)}{b - a}(x - a).$$

And this is *almost* the linear approximation formula, since $f'(a) \approx \frac{f(b)-f(a)}{b-a}$. As b gets closer to a, this will get closer and closer to being the linear approximation formula.

This line through $(a, f(a))$ and $(b, f(b))$ is a secant line. As b gets closer to a, then the two points the secant line goes through get closer together. When we take the limit, our line "goes through the same point twice". Thus it only touches the curve at one point—so it is a tangent line. Thus we see that the linear approximation to a function at a point a is the line tangent at that point a.

Example 2.45. Let $f(x) = \frac{x^3}{2} - x$. We can draw secant lines through the points $(0, f(0))$ and $(b, f(b))$, and see what happens as b gets closer to a. Below, we see the lines for $b = 1, 1/2, 1/10$, and then finally the tangent line given by the linear approximation formula.



We can see that each of the first three lines passes through two points, but as the points get closer and closer together, the secant lines better approximate the tangent line we see in the fourth picture.

We can see that this is, in fact, the same sort of question we asked earlier. The tangent line touches the function graph at one point, and is going in the "same direction" as the graph at that point. Thus it's the line that looks most like the point. So it *should* be the line that best approximates that function. And this is why the geometric tangent line quation is essentially the same as the algebraic linear approximation question.

Example 2.46 (Slope)**.** How can I think of the tangent line as a physical rate of change? If I'm thinking about the graph of a function, then the input to the function is a horizontal position, measured in inches (or some other unit of distance). And the output is a vertical position, also measured in inches. So $f(x)$ takes in inches and outputs inches.

The derivative $f'(x)$ will still take in inches. But if we compute the derivative $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, then the denominator is in inches and the numerator is also in inches. This makes the derivative technically unitelss—but in reality, it is measured in inches per inch.

And this has a clear physical interpretation! The slope of a line measures how many units the line goes up for each unit it goes over. Thus, it measures inches of horizontal position per inch of vertical position.

The second derivative $f''(x)$ will take in inches and output 1/inch, which is really inches per inch per inch. It tells us how much the slope, measured in inches per inch, changes if we move one inch horizontally.

Example 2.47.

2.9 Implicit Differentiation

We can push all these ideas about differentiation one step further. This time it makes the most sense to start with the geometric approach, and return to the other two later.

Let's start with a warmup example.

Example 2.48. Consider the curve defined by the equation $x^2 + y = 25$. Can we find a line tangent to this curve at the point $(3, 16)$?

This equation is not written as a function. Recall a function is a rule that takes an input and gives an output. And I haven't described a rule for you. But you can work out a rule that's hidden, or *implicit*, in this equation. A little rearranging gives us

$$y = 25 - x^2$$
$$\frac{dy}{dx} = -2x$$

and thus the derivative at $x = 3$ is -6. Then the equation for the tangent line is

$$y = 16 - 6(x - 3).$$

Now let's try a hairier example.

Example 2.49. Consider the equation $x^2 + y^2 = 25$, whose graph is a circle of radius 5. Can we find a tangent line to the curve when $x = 3$?

This is trickier, because we can't just reinterpret this equation as a function. We could try, and do something like

$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}.$$

But that $\pm$ symbol makes this not a real function. And derivatives are facts about *functions*. So what can we do?

We can't describe the whole circle as a function. But we can describe the top half of it as a function. The formula

$$y = \sqrt{25 - x^2}$$

gives us a perfectly fine function. We can differentiate this to get

$$y' = \frac{1}{2}(25 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{25 - x^2}},$$

and thus when $x = 3$ we get $y' = \frac{-3}{4}$. So the equation of our tangent line is

$$y = 4 - \frac{3}{4}(x - 3).$$

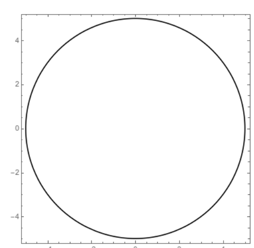


Figure 2.5: The circle $x^2 + y^2 = 25$.

But I have two problems with this. The first is simple: why did I take the positive square root and not the negative? It would have been just as valid to look at $y = -\sqrt{25 - x^2}$, and get a derivative of $3/4$ and a tangent line of $y = -4 + \frac{3}{4}(x - 3)$. I'd like a method that doesn't force me to make that choice.

The second, bigger problem is that this is too much work, and I'm lazy. The original equation is simple; I don't want to do a ton of work to turn it into something more complicated.

The key *idea* of our argument was that we can find a hidden function that sort of describes our equation. $y = \sqrt{25 - x^2}$ isn't the same as our equation, but as long as we're looking at positive y values, and don't worry too much about what's happening elsewhere, it gives us a good picture. The way I can be lazy now is just to assume that y is *some* function of x. But I won't worry about which function it is, and instead I'll just leave it as a named-but-unspecified function. (This is basically the whole trick of algebra: I don't know what this number is, so let's call it x and move on with our lives.)

If y is a function of x, now we get the equation

$$x^2 + (y(x))^2 = 25.$$

Each side of this equation is a function, and the two functions are the same. And that means that their derivatives are the same. I know the derivative of 25, and the derivative of x^2. I don't really know the derivative of $(y(x))^2$, since I don't even know what $y(x)$ is. But I'll just leave that unspecified again: by the chain rule, we know that

$$\frac{d}{dx}(y(x))^2 = 2y(x) \cdot y'(x).$$

Thus differentiating both sides of our original equation gives

$$2x + 2y(x)y'(x) = 0.$$

This doesn't give us the derivative of y exactly, but it does give us a formula! Rearranging this equation gives

$$\begin{aligned} 2y(x)y'(x) &= -2x \\ y'(x) &= \frac{-2x}{2y(x)} = \frac{-x}{y(x)}. \end{aligned}$$

And we get a formula for $y'(x)$ in terms of x and $y(x)$. This might seem like a problem, that I need two numbers to plug in and not just one. But this is actually revealing something deep about the problem. Remember that if $x = 3$, it's possible that $y = 4$ or $y = -4$. If I want to find the slope of the tangent line, I really do need to know which one I'm talking about.

And finally, we can say that if $x = 3$ and $y = 4$, then the derivative is $y'(x) = \frac{-3}{4}$. Which is, of course, what we got earlier.

Remark 2.50. There's one thing to beware of here. What if we look at the point $x = 5, y = 0$? Then our formula would have us dividing by 0, which isn't possible. We can see on the picture that the tangent line would be vertical. But it isn't a function, so the derivative there isn't well-defined.

Basically this is a failure of our idea, that if we zoom in on any point enough, its surroundings will look like a function. No matter how tight our focus, the curve near $(5, 0)$ will never look like the graph of a function, because it will always fail the vertical line test.

Example 2.51 (Folium of Descartes)**.** Let's consider a more complex equation, $x^3 + y^3 = 6xy$. This is known as the Folium of Descartes. We can compute the derivative of both sides:

$$\begin{aligned} \frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}(6xy) \\ 3x^2 + 3y^2\frac{dy}{dx} &= 6\left(y + x\frac{dy}{dx}\right) \\ (3y^2 - 6x)\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} \\ &= \frac{2y - x^2}{y^2 - 2x}. \end{aligned}$$

(Notice that I did in fact simplify at the end here. Because I'm about to use this formula to do a bunch of more computations, it's worth it to stop and simplify here to make my life easier.)

Now we can use this formula to find some tangent lines.

At the point $(3, 3)$ we compute that

$$\frac{dy}{dx} = \frac{6 - 9}{9 - 6} = -1$$

and thus the equation of the tangent line is $y - 3 = -(x - 3)$.

At the point $(0, 0)$, however, this doesn't actually give us a useful answer; the top and the bottom would both be zero. if you look at the picture in Figure 2.6, you see that there's not a clear tangentline there since the curve crosses itself. You can think of these "self-intersection" points as another way a function can fail to be differentiable, on our earlier list with corners, vertical tangents, and cusps.

We can also find second derivatives by extending this method. In this problem, we already know that

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}.$$

We can differentiate both sides of this. The derivative of the left side is just the derivative of the derivative, which is the second derivative. On the right we can use the quotient rule, so we get

$$\frac{d^2y}{dx^2} = \frac{\left(2\frac{dy}{dx} - 2x\right)(y^2 - 2x) - \left(2y\frac{dy}{dx} - 2\right)(2y - x^2)}{(y^2 - 2x)^2}.$$

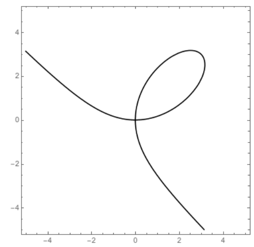


Figure 2.6: The folium of Descartes $x^3 + y^3 = 6xy$

This is okay, but it's a little unsatisfying; I'd like a formula purely in terms of x and y, and this formula also has the $\frac{dy}{dx}$ terms. But I can substitute in my earlier formula for $\frac{dy}{dx}$ and get

$$\frac{d^2y}{dx^2} = \frac{\left(2\frac{2y-x^2}{y^2-2x} - 2x\right)(y^2-2x) - \left(2y\frac{2y-x^2}{y^2-2x} - 2\right)(2y-x^2)}{(y^2-2x)^2}.$$

This is a little gross, but it does work. And we can compute now that the second derivative at $(3,3)$ is

$$\frac{d^2y}{dx^2} = \frac{(-2-6)(9-6)-(-6-2)(6-9)}{(9-6)^2} = \frac{-24-24}{9} = \frac{-16}{3}.$$

The exact number here is hard to interpret, but the fact that the second derivative is negative means that the slope of the tangent line decreases as we move to the right, which we can see on the graph.

Example 2.52. If $9x^2 + y^2 = 9$ then we have

$$\begin{aligned} 18x + 2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{9x}{y} \\ \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-\frac{9x}{y}\right) \\ &= -\frac{9y - 9x\frac{dy}{dx}}{y^2} \\ &= -\frac{9y - 9x(-\frac{9x}{y})}{y^2} \\ &= -\frac{9y + \frac{81x^2}{y}}{y^2} \end{aligned}$$

We see that at the point $(0,3)$ we have $y' = 0$ and $y'' = -3$. At the point $(\sqrt{5}/3, 2)$, then $y' = -\frac{3\sqrt{5}}{2}$ and $y'' = -\frac{18+\frac{45}{2}}{4}$.

Example 2.53. Find y'' if $x^6 + \sqrt[3]{y} = 1$. Then find the first and second derivatives at the point $(0, 1)$.

$$\begin{aligned} 6x^5 + \frac{1}{3}y^{-2/3}y' &= 0 \\ -18x^5y^{2/3} &= y' \\ -18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}y') &= y'' \\ -18(5x^4y^{2/3} + \frac{2}{3}x^5y^{-1/3}(-18x^5y^{2/3})) &= y'' \end{aligned}$$

Thus at $(0, 1)$, we have $y' = 0$ and $y'' = 0$. So the tangent line to the curve is horizontal at the point $(0, 1)$.

So far we've been looking at implicit differentiation as a geometric tool, to find tangent lines. But we can also use it algebraically, on relationships that apply to functions.

Example 2.54. Suppose we have some function f such that $8f(x) + x^2(f(x))^3 = 24$, and we want to find a linear approximation of f near $f(4) = 1$. (Say we've measured this experimentally and now want to understand or compute with the function). Then we have

$$\begin{aligned} \frac{d}{dx}\left(8f(x) + x^2(f(x))^3\right) &= \frac{d}{dx}24 \\ 8f'(x) + 2x(f(x))^3 + 3x^2(f(x))^2f'(x) &= 0 \\ 8f'(4) + 2 \cdot 4 \cdot 1^3 + 3 \cdot 4^2 \cdot 1^2 f'(4) &= 0 \\ 8f'(4) + 8 + 48f'(4) &= 0 \end{aligned}$$

and thus $f'(4) = -1/7$.

This leaves us with a question, though. We know $f(1)$; can we figure out the value of f at other points?

We have a derivative, so we can again compute a linear approximation. We get

$$f(x) \approx f'(4)(x-4) + f(4) = \frac{-1}{7}(x-4) + 1.$$

Thus we compute

$$f(5) \approx \frac{-1}{7}(5-4) + 1 = 1 + \frac{-1}{7} = \frac{6}{7} \approx .857.$$

Checking Mathematica, we see that the actual solution is .879. So we were pretty close.

2.10 Related Rates

Finally, let's apply a version of implicit differentiation to physical problems, or word problems.

It's good to take a moment here to talk about why we do work problems, and how to approach them. On a philosophical level, math does not tell us anything about the physical world. It only tells us that if certain properties hold, other things also have to be true. It's our job to take the aspect of the world we care about and translate it into math. Then we can see what the math implies, and hopefully that will still be true when translated back into the world.

Word problems are training for this process. We take verbal (or pictorial etc.) information, and try to turn it into a mathematical description. Then we see the mathematical consequences, and translate those back into a verbal description of physics.

So how do we approach this? **Checklist of steps for solving word problems:**

1. Draw a picture.
2. Think about what you expect the answer to look like. What is physically plausible?
3. Create notation, choose variable names, and label your picture.
   (a) Write down all the information you were given in the problem.
   (b) Write down the question in your notation.
4. Write down equations that relate the variables you have.
5. Abstractly: "solve the problem." Concretely differentiate your equation.
6. Plug in values and read off the answer.
7. Do a sanity check. Does you answer make sense? Are you running at hundreds of miles an hour, or driving a car twenty gallons per mile to the east?

Example 2.55. A spherical balloon is inflating at $12\text{cm}^3/\text{s}$. How quickly is the radius increasing when the radius is 3cm?

In a perfect world, we want to relate the rate at which the volume is changing to the rate at which the radius is changing. But we don't have any formulas lying around that relate those rates. What we *do* have are formulas that relate the levels: we can relate the "current" volume of the sphere to the "current" radius.

Specifically, we know that a sphere has volume $V = \frac{4}{3}\pi r^3$. Then we can differentiate both sides of that equation: using the chain rule, we compute that

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 4\pi r^2 \frac{dr}{dt}.$$

We know that $\frac{dV}{dr} = 12\text{cm}^3/\text{s}$, and $r = 3\text{cm}$. Substituting those facts in gives us

$$\begin{aligned} 12\text{cm}^3/\text{s} &= 4\pi(3\text{cm})^2 r' \\ r' &= \frac{1}{3\pi}\text{cm/s} \end{aligned}$$

So the radius is increasing by $1/3\pi$ cm per second.

Example 2.56. Suppose one car leaves Baltimore at noon, heading due north at 40 mph, and at 1 PM another car leaves Baltimore heading due west at 60 mph. At 2PM, how quicly is the distance between them increasing?

Write a for the distance the first car has traveled, and b for the distance the second car has traveled. We have that $a = 80\text{mi}, b = 60\text{mi}, a' = 40\text{mi/h}, b' = 60\text{mi/h}$. We want a formula that will relate the distances the cars have traveled to the distance between them; after drawing a picture we see this is the pythagorean theorem $a^2 + b^2 = c^2$, where c is the distance between the two cars.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 2c\frac{dc}{dt} &= 2a\frac{da}{dt} + 2b\frac{db}{dt}. \end{aligned}$$

We can use the pythagorean theorem to tell that $c = 100$, and thus we get

$$\begin{aligned} 2 \cdot (100\text{mi}) \cdot \frac{dc}{dt} &= 2 \cdot (80\text{mi}) \cdot (40\text{mi/h}) + 2 \cdot (60\text{mi}) \cdot (60\text{mi/h}) \\ 200\text{mi}\frac{dc}{dt} &= 6400\text{mi}^2/\text{h} + 7200\text{mi}^2/\text{h} \\ \frac{dc}{dt} &= 68\text{mi/h} \end{aligned}$$

so the distance between the cars is increasing at 68 mph.

The last thing we want to do is ask ourselves if this answer seems basically reasonable. The units are correct; we are expecting the distance to be increasing, so that checks out; and the size of the answer seems basically reasonable, because the cars are traveling at 40 mph and 60 mph and 68 is on the same scale as 40 and 60.

Example 2.57 (Recitation). A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end sliding out when the top is 12 feet from the ground?

Let h be the height of the ladder on the wall, and b be the distance of the foot of the ladder from the wall. Then $h = 12, h' = -1$, and $b = \sqrt{400 - 144} = 16$. We have

$$\begin{aligned} h^2 + b^2 &= 400 \\ 2hh' + 2bb' &= 0 \\ 2 \cdot 12 \cdot (-1) + 2 \cdot 16 \cdot b' &= 0 \\ b' &= \frac{24}{32} = 3/4 \end{aligned}$$

so the foot of the ladder is sliding away from the wall at 3/4 ft/s. Again, the direction of the sliding is correct (away from the wall), and the number seems plausible.

Example 2.58. A rectangle is getting longer by one inch per second and wider by two inches per second. When the rectangle is 5 inches long and 7 inches wide, how quickly is the area increasing?

We have $\ell = 5\text{in}, w = 7\text{in}, \frac{d\ell}{dt} = 1\text{in/s}, \frac{dw}{dt} = 2\text{in/s}$. We can relate all our quantities with the formula for the area of a rectangle: $A = \ell w$ relates the area, which we want to know about, to the length and width, which we do know about.

Taking a derivative gives us

$$\begin{aligned} \frac{dA}{dt} &= \ell \frac{dw}{dt} + w \frac{d\ell}{dt} \\ &= 5\text{in} \cdot 2\text{in/s} + 7\text{in} \cdot 1\text{in/s} \\ &= 17\text{in}^2/\text{s}. \end{aligned}$$

The units are right (the rate at which area is changing per second), and the direction is right (the area should be increasing, and this derivative is positive). It's really hard to see if the size is right using our intuition; people in general have bad intuition for the rate at which area changes in response to lengths.

But we can ask what would happen after a full second. One second later, we'd have $\ell = 6\text{in}$ and $w = 9\text{in}$ for a total area of 54in^2. This is an increase of 19in^2 over our starting area of 35in^2, and 17 is a pretty good approximation of 19.

As one final note, this is a problem we've basically seen before, in a different guise. The derivative of the area formula is just the product rule; we saw basically this same picture during the proof of the product rule in section 2.3.

Example 2.59. A street light is mounted at the top of a 15-foot-tall pole. A six-foot-tall man walks straight away from the pole at 5 feet per second. How fast is the distance between the pole and the tip of his shadow changing when he is forty feet from the pole?

There are actually two ways to set this up. The more obvious is to find an equation that will relate the length of the man's shadow to his distance from the pole, because we know how quickly the man is moving and we want to know how the shadow is changing. We see that we have a similar triangles situation, so if we say that m is the distance between the man and the pole, and s is the length of his shadow, we get the equation

$$\frac{6\text{ft}}{15\text{ft}} = \frac{s}{s+m}.$$

We could differentiate this using the quotient rule, but it's way easier if we collect terms first:

$$\begin{aligned} 6(s+m) &= 15s \\ 6m &= 9s \\ 6\frac{dm}{dt} &= 9\frac{ds}{dt} \\ \frac{2}{3} \cdot 5\text{ft/s} &= \frac{ds}{dt}. \end{aligned}$$

So the shadow is growing at a rate of $\frac{10}{3}$ft/s.

However, that is *not* the answer to the question I asked! I don't want to know how fast the shadow is growing; I want to know how fast the tip of the shadow is moving away from the pole. So I need to add $\frac{ds}{dt}$ the rate at which the shadow is growing, to the rate at which the base of the shadow is moving away from the pole, which is $\frac{dm}{dt}$. So my final answer is that the tip of the shadow is moving away from the pole at $(5 + 10/3)\text{ft/s} = \frac{25}{3}\text{ft/s}$.

But once I realize that $\frac{ds}{dt}$ isn't actually the thing I need to know, maybe I can set the question up differently. Let m be the distance between the man and the pole, and let L be the distance from the pole to the tip of the shadow—which is the thing that we actually care about. WE can make the same similar triangles equation, but this time we get

$$\begin{aligned}
\frac{6\text{ft}}{15\text{ft}} &= \frac{L-m}{L} \\
6L &= 15(L-m) \\
15m &= 9L \\
15\frac{dm}{dt} &= 9\frac{dL}{dt} \\
15 \cdot 5sift/s &= 9\frac{dL}{dt} \\
\frac{dL}{dt} &= \frac{25}{3}\text{ft/s}
\end{aligned}$$

and thus the distance between the pole and the tip of the shadow is increasing at $\frac{25}{3}$ feet per second.

Example 2.60. An inverted conical water tank with radius 2m and height 4m is being filled with water at a rate of $2\text{m}^3/\text{min}$. How fast is the water rising when the water level is 3m high in the tank?

We know that we want to relate the height of the water h to the volume of water V. The obvious equation to use here is the formula for the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h.$$

But this has a problem; in addition to V and h, this equation includes an r, which we don't know anything about. (The problem gives us a radius, but it's the radius of the *tank*, not the water filling the tank.)

The more naive approach is to plunge boldly ahead. We can take a derivative, and we get

$$\begin{aligned}
\frac{dV}{dt} &= \frac{\pi}{3}\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right) \\
2\text{m}^3/\text{min} &= \frac{\pi}{3}\left(2r\frac{dr}{dt} \cdot 3\text{m} + r^2\frac{dh}{dt}\right),
\end{aligned}$$

but we still don't have values for r or $\frac{dr}{dh}$.

We need a new equation, that will relate r to something we already know about. But we know that the water is *in the conical tank*, and should have the same shape. In particular, the sides of our cones are similar triangles! The ratio of the radius of the tank to the height of the tank must be the same as the ratio of the radius of the water to the height of the water. So we get

$$\begin{aligned}\frac{2\text{m}}{4\text{m}} &= \frac{r}{h}\\ r &= \frac{h}{2}\left(3/2\text{m}\right)\\ \frac{dr}{dt} &= \frac{1}{2}\frac{dh}{dt}.\end{aligned}$$

We can substitute this back into our original equation, and we get

$$\begin{aligned}2\text{m}^3/\text{min} &= \frac{\pi}{3}\left(2\cdot\frac{3}{2}\text{m}\cdot\frac{1}{2}\frac{dh}{dt}\cdot 3\text{m} + \left(\frac{3}{2}\text{m}\right)^2\frac{dh}{dt}\right)\\ &= \frac{\pi}{3}\left(\frac{9}{2}\frac{dh}{dt}\text{m}^2 + \frac{9}{4}\frac{dh}{dt}\text{m}^2\right)\\ &= \frac{9\pi}{4}\text{m}^2\frac{dh}{dt}\\ \frac{dh}{dt} &= \frac{8}{9\pi}\text{m/min}.\end{aligned}$$

So we conclude that the water level is rising at $\frac{8}{9\pi}$ meters per minute.

However, while that worked, it was a huge mess algebraically. If we're smart we can do this much more easily. We start with the volume equation

$$V = \frac{1}{3}\pi r^2 h.$$

At this point, we can *notice* that the r will be a problem, so we get rid of it now. We make the similar triangles and see that

$$\begin{aligned}\frac{2\text{m}}{4\text{m}} &= \frac{r}{h}\\ r &= \frac{h}{2}\end{aligned}$$

and substituting that back into the original equation gives

$$\begin{aligned}V &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3\\ \frac{dV}{dt} &= \frac{\pi}{12}3h^2\frac{dh}{dt}\\ 2\text{m}^3/\text{min} &= \frac{\pi}{4}\cdot(3\text{m})^2\frac{dh}{dt}\\ \frac{8}{9\pi}\text{m/min} &= \frac{dh}{dt}.\end{aligned}$$

So we conclude that the water level is rising at $\frac{8}{9\pi}$ meters per minute.

Example 2.61. A lighthouse is located three kilometers away from the nearest point P on shore, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline 1 kilometer from P?

Let's say the angle of the light away from P is θ, and the distance from P is d. Then we have $d = 1$ and $\theta' = 8\pi$ (in radians per minute). We also have the relationship that $\tan\theta = \frac{d}{3}$.

Taking the derivative gives us $\sec^2(\theta) \cdot \theta' = d'/3$. We need to work out $\sec^2(\theta)$, but looking at our triangle we see that the adjacent side is length 3 and the hypotenuse is length $\sqrt{10}$ (by the Pythagorean theorem), so we have $\sec^2(\theta) = (\sqrt{10}/3)^2 = 10/9$.

Thus we have $d' = 3\sec^2(\theta) \cdot 8\pi = \frac{80\pi}{3}$ kilometers per second.

Example 2.62 (Recitation). A kite is flying 100 feet over the ground, moving horizontally at 8 ft/s. At what rate is the angle between the string and the ground decreasing when 200ft of string is let out?

Call the distance between the kite-holder and the kite d and the angle between the string and the ground θ. When the length of string is 200 then $d = \sqrt{200^2 - 100^2} = 100\sqrt{3}$. We have that $d' = 8$ (since the angle is decreasing, the kite must be getting farther away). And finally we have the relationship $\tan\theta = \frac{100}{d}$ by the definition of tan in terms of triangles. Then we have

$$\begin{aligned} \tan\theta &= 100d^{-1} \\ \sec^2(\theta)\theta' &= -100d^{-2}d' \\ \theta' &= \frac{-100 \cdot 8\cos^2(\theta)}{d^2}. \end{aligned}$$

We see that $\cos(\theta) = \frac{100\sqrt{3}}{200} = \sqrt{3}2$, so we have

$$\theta' = \frac{-100 \cdot 8 \cdot 3/4}{(100\sqrt{3})^2} = -\frac{8}{100 \cdot 4} = \frac{-1}{50}.$$

So the angle between the string and the ground is decreasing at a rate of $1/50$ per second. (Note: radians are unitless!)