Math 1231 Spring 2023
Single-Variable Calculus 1 Section 12
Mastery Quiz 10
Due Tuesday, April 4

This week's mastery quiz has three topics. Everyone should submit work on S6. If you already have a 4/4 on M4, you should not submit it. But if Blackboard doesn't say you're at a 4/4, then you should submit again for another try. If you already have a 2/2 on S5, you don't need to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Optimization
- Secondary Topic 5: Curve Sketching
- Secondary Topic 6: Physical Optimization

Name:

Recitation Section:

M4: Extrema and Optimization

(a) Classify all the critical points and relative extrema of $h(x) = x^3/(x+1)$. (For each critical point, tell me whether it is a relative maximum, a relative minimum, or neither.)

Solution: We have

$$h'(x) = \frac{3x^2(x+1) - x^3}{(x+1)^2} = \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} = \frac{x^2(2x+3)}{(x+1)^2}$$

The critical points are thus at 0 and at $-3/2$, and a fake one at -1. We make a chart:

	x^2	$2x+3$	$(x+1)^{-2}$	$h'(x)$
$x < -3/2$	+	−	+	−
$-3/2 < x < -1$	+	+	+	+
$-1 < x < 0$	+	+	+	+
$0 < x$	+	+	+	+

This tells us that we have a local minimum at $x = -3/2$, and no other extrema. We compute $h(-3/2) = -27/8/(-1/2) = 27/4$, so the sole local minimum is $(-3/2, 27/4)$.

Alternatively we could try the second derivative test. The second derivative is

$$h''(x) = \frac{2x(x^2+3x+3)}{(x+1)^3}.$$

We compute that

$$h''(0) = 0$$
$$h''(-3/2) = \frac{-3(9/4 - 9/2 + 3)}{(-1/2)^3} = 24(3/4) = 18 > 0.$$

This tells us we have a relative minimum at $-3/2$, but doesn't help us with the critical point at 0. Note: we *cannot* conclude from this that we don't have an extremum at 0, even though that is in fact true!

(b) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x-4)(x-1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$f(0) = 7 \qquad\qquad f(1) = 14$$
$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$
$$f(5) = 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

S5: Curve Sketching

Sketch the graph of $f(x) = x^5 - 5x^4 + 5x^3 = x^3(x^2 - 5x + 5)$. We have $f'(x) = 5x^2(x-3)(x-1)$ and $f''(x) = 10x(2x^2 - 6x + 3)$.

You should discuss the domain, limits at infinity, critical points, intervals of increase and decrease, concavity, and possible points of inflection.

Solution: The domain of f is all reals.

There are roots at 0 and at $5/2 \pm \sqrt{5}/2$.

We see that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

The critical points are $0, 1, 3$. We compute $f(0) = 0, f(1) = 1, f(3) = -27$.

For increase and decrease we make a chart:

	$5x^2$	$(x-3)$	$(x-1)$	$f'(x)$
$x < 0$	+	−	−	+
$0 < x < 1$	+	−	−	+
$1 < x < 3$	+	−	+	−
$3 < x$	+	+	+	+

Thus f is increasing on $(-\infty, 1)$ and on $(3, +\infty)$, and is decreasing on $(1, 3)$.

The possible points of inflection are 0 and $\frac{6\pm\sqrt{36-24}}{4} = \frac{3\pm\sqrt{3}}{2}$. We can make a chart:

	$10x$	$2x^2 - 6x + 3$	$f'(x)$
$x < 0$	−	+	−
$0 < x < (3-\sqrt{3})/2$	+	+	+
$(3-\sqrt{3})/2 < x < (3+\sqrt{3}))/2$	+	−	−
$(3+\sqrt{3})/2 < x$	+	+	+

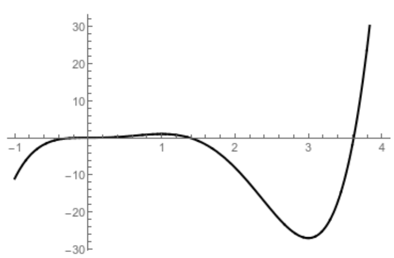


Figure 1: Graph of $f(x)$

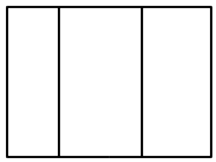
S6: Physical Optimization

We wish to build a rectangular pen with two parallel internal partitions, using 1000 feet of fencing. We want to maximize the total area of the pen.

(a) What is your objective function, and why?

(b) What constraint equation(s) can you use?

(c) What dimensions maximize the total area of the pen?



Solution: Our objective function is $A = \ell w$, because this is the area we want to maximize. We see also that we have the constraint $2\ell + 4w = 1000$ so we can write $\ell = 500 - 2w$, and thus we have

$$A = (500 - 2w)w = 500w - 2w^2$$
$$A' = 500 - 4w$$

has a critical point when $w = 125$.

We can see this is a maximum using the extreme value theorem: the function is defined on the interval $[0, 250]$, and $A(0) = A(250) = 0$.

Or we can use the first derivative test; we see that $A'(w) < 0$ when $w > 125$ and $A'(w) > 0$ when $w < 125$, so A has a local maximum at $w = 125$.

Or we can use the second derivative test. $A'' = -4 < 0$ so we have a local maximum.

Thus the pen is maximized with width 125 and length 250. (The maximum area, which I didn't ask for, is 31250 square feet.)