Math 1231 Spring 2023
Single-Variable Calculus 1 Section 12
Mastery Quiz 11
Due Tuesday, April 11

This week's mastery quiz has three topics. Everyone should submit work on S7. If you already have a 4/4 on M4, you should not submit it. But if Blackboard doesn't say you're at a 4/4, then you should submit again for another try. If you already have a 2/2 on S6, you don't need to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Optimization
- Secondary Topic 6: Physical Optimization
- Secondary Topic 7: Riemann Sums

Name:

Recitation Section:

M4: Extrema and Optimization

(a) Classify the critical points and relative extrema of $h(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$.

Solution: We have

$$h'(x) = \cos(x) - \sin(x)$$

so $h'(x)$ is defined everywhere, and has critical points where $\cos(x) = \sin(x)$. This happens when $x = \pi/4, 5\pi/4, 9\pi/4, \cdots = \pi/4 + n\pi$. We only need to care about $\pi/4$ and $5\pi/4$.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$\begin{aligned} h''(x) &= -\sin(x) - \cos(x) \\ h''(\pi/4) &= -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0 \\ h''(5\pi/4) &= \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2} > 0. \end{aligned}$$

Thus h has a local maximum at $\pi/4$ and has a local minimum at $5\pi/4$.

For the first derivative test we make a chart:

	$h'(x)$
$0 < x < \pi/4$	$+$
$\pi/4 < x < 5\pi/4$	$-$
$5\pi/4 < x < 2\pi$	$+$

so h has a relative maximum at $\pi/4$ and a relative minimum at $5\pi/4$.

(b) Find the absolute extrema of $f(x) = x^3 + x^2 - 5x$ on $[-1, 2]$, and justify your claim that they are extrema.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 3x^2 + 2x - 5 = (3x + 5)(x - 1)$ is defined everywhere and has roots at $-5/3$ and 1, so the critical points are $-5/3, 1$. We can ignore $-5/3$ because it isn't in the interval, so we need to evaluate f at $-1, 1, 2$.

$$\begin{aligned} f(-1) &= 5 \\ f(1) &= -3 \\ f(2) &= 2 \end{aligned}$$

So the absolute minimum is -3 at 1 , and the absolute maximum is 5 at -1.

S6: Physical Optimization

Suppose that a company that produces and sells x units of a product makes a revenue of $R(x) = 260x - 9x^2/10$ and has costs given by $C(x) = 1000 + 100x + x^2/10$. What is the maximum profit that can be made (where profit is revenues minus costs)?

Solution: Our profit function is $P(x) = R(x) - C(x) = -1000 + 160x - x^2$. Then

$$P'(x)160 - 2x$$
$$160 = 2x$$
$$80 = x$$

We can check that this is truly a maximum by the second derivative: $P''(x) = -2 < 0$ so we have a local maximum.

Or we can see that $P'(x) > 0$ when $x < 80$ and $P'(x) < 0$ when $x > 80$, so the function is increasing until 80 and decreasing after.

The profit at this quantity is

$$P(80) = -1000 + 160(80) - (80)^2 = -1000 + 12800 - 6400 = 5400.$$

Secondary Topic 7: Riemann Sums

Let $f(x) = x^2 - x$ be defined on the interval $[-3, 0]$.

(a) Approximate the area under the curve of the function using three rectangles and right endpoints.

(b) Approximate the area under the curve of the function using three rectangles and left endpoints.

(c) Write a formula for R_n, the estimate using n rectangles and right endpoints, as a summation of n terms.

(d) Use your answer in part (c) to find a closed-form formula for R_n. (This formula should not have a summation sign or be given as a sum of n terms.)

(e) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_3 = 1 \cdot f(-2) + 1 \cdot f(-1) + 1 \cdot f(0) = 6 + 2 + 0 = 8.$

(b) $L_3 = 1 \cdot f(-3) + 1 \cdot f(-2) + 1 \cdot f(-1) = 12 + 6 + 2 = 20.$

(c)

$$R_n = \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n}\left((3i/n - 3)^2 - (3i/n - 3)\right)$$

(d)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n}\left((3i/n-3)^2 - (3i/n-3)\right) \\
&= \sum_{i=1}^{n} \frac{3}{n}\left(\frac{9i^2}{n^2} - \frac{18i}{n} + 9 - \frac{3i}{n} + 3\right) \\
&= \sum_{i=1}^{n} \frac{27i^2}{n^3} - \frac{63i}{n^2} + \frac{36}{n} \\
&= \frac{27}{n^3}\sum_{i=1}^{n} i^2 - \frac{63}{n^2}\sum_{i=1}^{n} i + \frac{36}{n}\sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n
\end{aligned}$$

(e) We can compute

$$\begin{aligned}
\lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2}\frac{n(n+1)}{2} + \frac{36}{n}n \\
&= \lim_{n\to+\infty} \frac{27\cdot 1(1+1/n)(2+1/n)}{6} - \frac{63\cdot 1(1+1/n)}{2} + 36 \\
&= 9 - \frac{63}{2} + 36 = \frac{27}{2}.
\end{aligned}$$