Math 1231: Single-Variable Calculus 1
George Washington University   Spring 2023
Recitation 11

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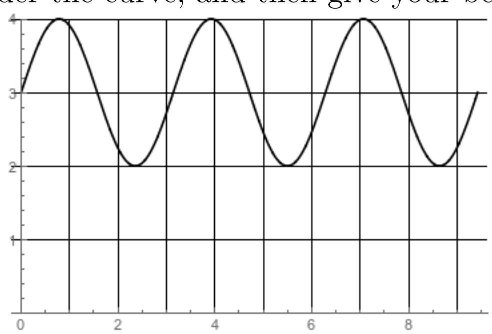
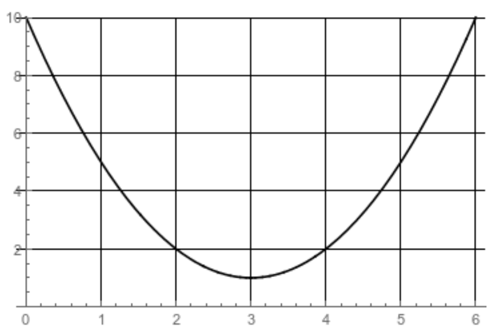
April 7, 2023

Problem 1. For the following curves, find an upper bound and a lower bound for the area under the curve, and then give your best estimate for the actual area.



(between 0 and 9; ignore the trailing bit off the right edge)

Solution: Upper bound: 36;
Lower bound: 18;
exact value: $9\pi \approx 28$



(between 0 and 6; ignore the trailing bit off the right edge)

Solution: Upper bound: 36;
Lower bound: 12;
exact value: 24

Problem 2. Consider the function $f(x) = \sqrt{1 - x^2}$ between $x = 0$ and $x = 1$.

(a) Estimate the area using two rectangles with right endpoints. Is this an upper bound, a lower bound, or neither?

(b) Estimate the area using two rectangles with left endpoints. Is this an upper bound, lower bound, or neither?

(c) Find an upper bound using four rectangles.

(d) Find a lower bound using four rectangles.

(e) Can you guess what the area under the curve is exactly? (Hint: what does the graph look like?)

Solution:

(a)
$$R_2 = \frac{1}{2}\left(\sqrt{1-1/4}+0\right) = \frac{\sqrt{3}}{4} \approx .43.$$
This is a lower bound, since the function is decreasing and the right endpoint is always the lowest point in the interval.

(b)
$$L_2 = \frac{1}{2}\left(1+\sqrt{1-1/4}\right) = \frac{4+\sqrt{3}}{4} \approx .93.$$
This is an upper bound, since the function is decreasing and the left endpoint is always the highest point in the interval.

(c)
$$L_4 = \frac{1}{4}\left(1+\sqrt{1-1/16}+\sqrt{1-1/4}+\sqrt{1-9/16}\right) = \frac{4+\sqrt{15}+2\sqrt{3}+\sqrt{7}}{16} \approx .87.$$

(d)
$$R_4 = \frac{1}{4}\left(\sqrt{1-1/16}+\sqrt{1-1/4}+\sqrt{1-9/16}+0\right) = \frac{\sqrt{15}+2\sqrt{3}+\sqrt{7}}{16} \approx .62.$$

(e) This is the graph of a quarter circle—in fact, the upper-right quadrant of the unit circle—so the area should be $\pi/4 \approx .79$.

Problem 3. Consider the function $g(x) = x^3$ between $x = 0$ and $x = 1$.

(a) Estimate the area using two rectangles with right endpoints. Is this an upper bound, a lower bound, or neither?

(b) Estimate the area using two rectangles with left endpoints. Is this an upper bound, lower bound, or neither?

(c) Find an upper bound using four rectangles.

(d) Find a lower bound using four rectangles.

(e) Find a formula using right endpoints to estimate the area using n rectangles, in summation form.

(f) Use your summation rules to get a closed-form formula, with no summation signs in it.

(g) Take a limit to find the exact area under this curve. (Use your summation rules!)

Solution:

(a)

$$R_2 = \frac{1}{2}\left(\frac{1}{2}^3 + 1^3\right) = \frac{9}{16}$$

This is an upper bound, since the function is increasing.

(b)

$$L_2 = \frac{1}{2}\left(0^3 + \frac{1}{2}^3\right) = \frac{1}{16}.$$

This is a lower bound, since the function is decreasing.

(c)

$$R_4 = \frac{1}{4}\left(\frac{1}{4}^3 + \frac{1}{2}^3 + \frac{3}{4}^3 + 1^3\right) \approx .39.$$

(d)

$$L_4 = \frac{1}{4}\left(0 + \frac{1}{4}^3 + \frac{1}{2}^3 + \frac{3}{4}^3 + \right) \approx .14$$

(e)

$$R_n = \sum_{i=1}^{n} f\left(0 + i\frac{1-0}{n}\right) \cdot \frac{1-0}{n} = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \cdot \frac{1}{n}.$$

$$\begin{aligned} R_n &= \frac{1}{n}\sum_{i=1}^{n} \frac{i}{n}^3 = \frac{1}{n}\frac{1}{n^3}\sum_{i=1}^{n} i^3 \\ &= \frac{1}{n^4}\left(\frac{(n)(n+1)}{2}\right)^2 = \frac{n^4 + 2n^3 + n^2}{4n^4}. \end{aligned}$$

(f)

$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} \frac{n^2 + 2n + 1}{4n^2} = \lim_{n\to+\infty} \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4} = \frac{1}{4}.$$

Problem 4. Let $h(x) = x^2 + 3x - 2$ between $x = -2$ and $x = 2$.

(a) Write down a summation formula for R_n.

(b) Use summation rules to get a closed-form formula.

(c) Take a limit to compute $\int_{-2}^{2} x^2 + 3x - 2\, dx$.

(d) What does this refer to geometrically? Does that make sense?

Solution:

(a) We have $\Delta x = \frac{b-a}{n} = \frac{4}{n}$, so we get the formula

$$\begin{aligned} R_n &= \sum_{i=1}^{n} h\left(-2 + i\frac{4}{n}\right) \cdot \frac{4}{n} \\ &= \sum_{i=1}^{n} \left(\left(\frac{4i}{n} - 2\right)^2 + 3\left(\frac{4i}{n} - 2\right) - 2\right)\frac{4}{n} \\ &= \sum_{i=1}^{n} \left(\frac{16i^2}{n^2} - \frac{16i}{n} + 4 + \frac{12i}{n} - 6 - 2\right)\frac{4}{n} \\ &= \sum_{i=1}^{n} \frac{64i^2}{n^3} - \frac{16i}{n^2} - \frac{16}{n}. \end{aligned}$$

(b)

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \frac{64i^2}{n^3} - \frac{16i}{n^2} - \frac{16}{n} \\ &= \sum_{i=1}^{n} \frac{64i^2}{n^3} - \sum_{i=1}^{n} \frac{16i}{n^2} - \sum_{i=1}^{n} \frac{16}{n} \\ &= \frac{64}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^2}\sum_{i=1}^{n} i - \frac{16}{n}\sum_{i-1}^{n} 1 \\ &= \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^2}\frac{n(n+1)}{2} - \frac{16}{n}\cdot n \\ &= \frac{32(2n^2+3n+2)}{3n^2} - \frac{8n+8}{n} - 16 \end{aligned}$$

(c)

$$\begin{aligned} \int_{-2}^{2} x^2 + 3x - 2\, dx &= \lim_{n\to+\infty} R_n = \lim_{n\to+\infty} \frac{32(2n^2+3n+2)}{3n^2} - \frac{8n+8}{n} - 16 \\ &= \frac{64}{3} - 8 - 16 = \frac{64}{3} - \frac{72}{3} = \frac{-8}{3}. \end{aligned}$$

(d) This is the signed area under the curve: the amount of area above the x-axis minus the amount of area below the x-axis. In particular, it's *not* the area, because it's a negative number and areas can't be negative.