Math 1231 Spring 2023
Single-Variable Calculus 1 Section 12
Mastery Quiz 12
Due Tuesday, April 18

This week's mastery quiz has three topics. Everyone should submit work on S7. If you already have a 4/4 on M4, you should not submit it. But if Blackboard doesn't say you're at a 4/4, then you should submit again for another try. If you already have a 2/2 on S6, you don't need to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 5: Integration
- Secondary Topic 7: Riemann Sums

Name:

Recitation Section:

Major Topic 5: Integration

(a) Let $F(x) = \int_2^{\sqrt{x^2+1}} t\sin(t)\,dt$. What is $F'(x)$?

Solution: If we set $F_1(x) = \int_2^x t\sin(t)\,dt$ then $F_1'(x) = x\sin(x)$, so

$$\frac{d}{dx}F(x) = \frac{d}{dx}F_1(\sqrt{x^2+1}) = \sqrt{x^2+1}\sin(\sqrt{x^2+1})\frac{2x}{2\sqrt{x^2+1}}.$$

(b) Find $\int \cos(x) + 2x\,dx$.

Solution: $\int \cos(x) + 2x\,dx = \sin(x) + x^2 + C$.

(c) Compute $\int_{-2}^{4} x^3 - 3x\,dx =$

Solution:

$$\int_{-2}^{4} x^3 - 3x\,dx = \frac{x^4}{4} - \frac{3x^2}{2}\Big|_{-2}^{4}$$
$$= 64 - 4 - 24 + 6 = 42.$$

Secondary Topic 7: Riemann Sums

Let $f(x) = 2x + 4x^2$ be defined on the interval $[0, 2]$.

(a) Approximate the area under the curve of the function using four rectangles and right endpoints.

(b) Approximate the area under the curve of the function using four rectangles and left endpoints.

(c) Write a formula for R_n, the estimate using n rectangles and right endpoints, as a summation of n terms.

(d) Use your answer in part (c) to find a closed-form formula for R_n. (This formula should not have a summation sign or be given as a sum of n terms.)

(e) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_4 = 1/2 \cdot f(1/2) + 1/2 \cdot f(1) + 1/2 \cdot f(3/2) + 1/2 \cdot f(2) = \frac{1}{2}(2 + 6 + 12 + 20) = 20$

(b) $L_4 = 1/2 \cdot f(0) + 1/2 \cdot f(1/2) + 1/2 \cdot f(1) + 1/2 \cdot f(3/2) = \frac{1}{2}(0 + 2 + 6 + 12) = 10$

(c)

$$R_n = \sum_{i=1}^{n} \frac{2}{n} f\left(0 + i\frac{2}{n}\right) = \sum_{i=1}^{n} \frac{2}{n}\left(2(2i/n) + 4(2i/n)^2\right)$$

(d)

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \frac{2}{n}\left(2(2i/n) + 4(2i/n)^2\right) \\ &= \sum_{i=1}^{n} \frac{2}{n}\left(\frac{4i}{n} + \frac{16i^2}{n^2}\right) \\ &= \sum_{i=1}^{n} \frac{8i}{n^2} + \frac{32i^2}{n^3} \\ &= \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^2}\frac{n(n+1)}{2} + \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6} \end{aligned}$$

(e) We can compute

$$\begin{aligned} \lim_{n\to+\infty} R_n &= \lim_{n\to+\infty} \frac{8}{n^2}\frac{n(n+1)}{2} + \frac{32}{n^3}\frac{n(n+1)(2n+1)}{6} \\ &= 4 + \frac{32}{3} = \frac{44}{3}. \end{aligned}$$