Math 1231 Spring 2023
Single-Variable Calculus 1 Section 12
Mastery Quiz 13
Due Tuesday, April 25

This week's mastery quiz has two topics. Everyone should submit both.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 5: Integration
- Major Topic 6: Applications of Integration

Name:

Recitation Section:

Major Topic 5: Integration

(a) Compute $\displaystyle\int \cos(5x+3)\,dx$.

Solution: Set $u = 5x+3$ so $du = 5dx$ and $dx = du/5$. Then

$$\int \cos(5x+3)\,dx = \int \cos(u)\frac{du}{5} = \frac{\sin(u)}{5} + C = \frac{1}{5}\sin(5x+3) + C.$$

(b) By explicitly changing the bounds of integration, compute $\displaystyle\int_1^2 x^5\sqrt{x^3+8}\,dx$.

Solution: We take $u = x^3+8$, so we have $du = 3x^2\,dx, dx = \frac{du}{3x^2}$, and we have $g(1) = 9$ and $g(2) = 16$. Then we compute

$$\begin{aligned}\int_1^2 x^5\sqrt{x^3+8}\,dx &= \int_9^{16} x^5\sqrt{u}\frac{du}{3x^2} \\ &= \frac{1}{3}\int_9^{16} x^3\sqrt{u}\,du = \frac{1}{3}\int_9^{16}(u-8)\sqrt{u}\,du \\ &= \frac{1}{3}\int_9^{16} u^{3/2} - 8u^{1/2}\,du \\ &= \frac{1}{3}\left(\frac{u^{5/2}}{5/2} - \frac{8u^{3/2}}{3/2}\right)\Bigg|_9^{16} \\ &= \frac{1}{3}\left(\left(\frac{2\cdot 4^5}{5} - \frac{8\cdot 2\cdot 4^3}{3}\right) - \left(\frac{2\cdot 3^5}{5} - \frac{8\cdot 2\cdot 3^3}{3}\right)\right) \\ &= \frac{2048}{15} - \frac{1024}{9} - \frac{162}{5} + 48 = \frac{1726}{45} = 38.355\ldots.\end{aligned}$$

(c) Compute $\displaystyle\int_0^2 \frac{x^2}{\sqrt{5x^3+9}}\,dx$.

Solution: We take $u = 5x^3+9$ so $du = 15x^2\,dx$. That gives us $u(0) = 9$ and $u(2) = 49$. Then we have

$$\begin{aligned}\int_0^2 \frac{x^2}{\sqrt{5x^3+9}}\,dx &= \int_9^{49}\frac{x^2}{\sqrt{u}}\frac{du}{15x^2} = \frac{1}{15}\int_9^{49} u^{-1/2}\,du \\ &= \frac{1}{15}\cdot(2u^{1/2})\Big|_9^{49} = \frac{2}{15}\cdot 7 - \frac{2}{15}\cdot 3 = \frac{8}{15}.\end{aligned}$$

Alternatively, you could compute

$$\begin{aligned}\int \frac{x^2}{\sqrt{5x^3+9}}\,dx &= \int \frac{x^2}{\sqrt{u}}\frac{du}{15x^2} = \frac{1}{15}\int u^{-1/2}\,du \\ &= \frac{1}{15}\cdot 2u^{1/2} + C = \frac{2}{15}\sqrt{5x^3+9} + C\end{aligned}$$

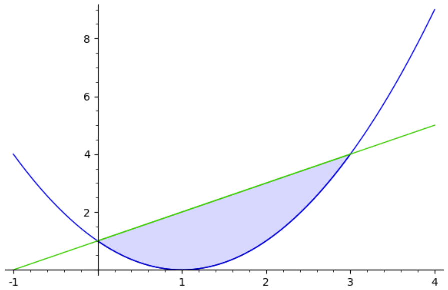
and so

$$\int_0^2 \frac{x^2}{\sqrt{5x^3+9}}\,dx = \frac{2}{15}\sqrt{5x^3+9}\Big|_0^2 = \frac{2}{15}\sqrt{49} - \frac{2}{15}\sqrt{9} = \frac{8}{15}.$$

Major Topic 6: Applications of Integrals

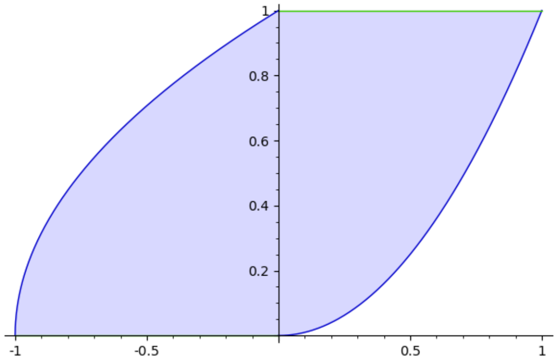
(a) Sketch the region bounded by the curves $y = (x-1)^2$ and $y = x+1$. Find the area of the region.

Solution: We sketch the region, and see that it will be much easier to integrate with respect to x. Setting the two equations equal, we see the curves intersect when $x + 1 = x^2 - 2x + 1$, and thus when $0 = x^2 - 3x = x(x-3)$, and so when $x = 0, 3$. So the x coordinates vary from 0 to 3.



$$\begin{aligned} A &= \int_0^3 x + 1 - (x-1)^2\,dx = \int_0^3 x + 1 - x^2 + 2x - 1\,dx \\ &= \int_0^3 -x^2 + 3x\,dx = -x^3/3 + 3x^2/2\Big|_0^3 \\ &= -9 + 27/2 = 9/2. \end{aligned}$$

(b) Sketch the region bounded by the curves $x = y^2 - 1$, $y = 0$, $y = 1$, and $x = \sqrt{y}$, and find the area of that region.



Solution:

We really want to integrate this with respect to y. So we have

$$\int_0^1 \sqrt{y} - (y^2 - 1)\,dy = \int 1 + \sqrt{y} - y^2\,dy$$
$$= y + \frac{2}{3}y^{3/2} - \frac{y^3}{3}\Big|_0^1 = 1 + \frac{2}{3} - \frac{1}{3} = \frac{4}{3}.$$

If we really want to, though, we can integrate with respect to x. We compute

$$A = \int_{-1}^0 \sqrt{x+1}\,dx + \int_0^1 1 - x^2\,dx$$
$$= \frac{2}{3}(x+1)^{3/2}\Big|_{-1}^0 + x - \frac{x^3}{3}\Big|_0^1$$
$$= \frac{2}{3} - 0 + 1 - \frac{1}{3} - 0 + 0 = \frac{4}{3}.$$