Math 1231: Single-Variable Calculus 1
George Washington University Spring 2023
Recitation 13

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April 21, 2023

Problem 1. Compute the following integrals:

(a) $\displaystyle\int \sqrt{3x-4}\,dx.$

Solution: We can take $u = 3x - 4$, which is the inside function. Then $du = 3\,dx$ so we have

$$\begin{aligned}\int \sqrt{3x-4}\,dx &= \int \sqrt{u}\frac{du}{3} = \frac{1}{3}\int \sqrt{u}\,du \\ &= \frac{1}{3}\int u^{1/2}\,du = \frac{1}{3}\frac{u^{3/2}}{3/2} + C \\ &= \frac{1}{3}\frac{(3x-4)^{3/2}}{3/2} + C.\end{aligned}$$

(b) $\displaystyle\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx.$

Solution: Set $u = \sqrt{x}$, and $du = \frac{dx}{2\sqrt{x}}$. Thus

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}} = \int 2\sin(u)\,du = -2\cos(u) + C = -2\cos(\sqrt{x}) + C.$$

(c) $\displaystyle\int x\sqrt{x+1}\,dx.$

Solution: Take $u = x+1$, $du = 1 \cdot dx$. Then

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du = \int u^{3/2} - u^{1/2}\,du \\ &= \frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} + C \\ &= \frac{2(x+1)^{5/2}}{5} - \frac{2(x+1)^{3/2}}{3} + C.\end{aligned}$$

Problem 2. (a) Compute $\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx$ using a u-substitution and explicitly changing the bounds of integration.

(b) Now compute the indefinite integral $\int \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx$.

(c) Use your answer in part (b) to compute $\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx$ again. How does this compare to what you did in part (a)?

Solution:

(a) We take $u = 2x^3 - 7x + 14$, so $du = (6x^2-7)\,dx$. Then $u(1) = 9$ and $u(2) = 16$. So we have

$$\begin{aligned}\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx &= \int_9^{16} \frac{6x^2-7}{\sqrt{u}} \frac{du}{6x^2-7} \\ &= \int_9^{16} u^{-1/2}\,du = 2u^{1/2}\Big|_9^{16} = 8-6 = 2.\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx &= \int \frac{6x^2-7}{\sqrt{u}} \frac{du}{6x^2-7} \\ &= \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{2x^3-7x+14} + C.\end{aligned}$$

(c)

$$\begin{aligned}\int_1^2 \frac{6x^2-7}{\sqrt{2x^3-7x+14}}\,dx &= 2\sqrt{2x^3-7x+14}\Big|_1^2 \\ &= 2\sqrt{16} - 2\sqrt{9} = 8-6 = 2.\end{aligned}$$

Notice that we wind up not only with the same answer, but the same final calculation, as in part (a); we're plugging the same values in to the same function $2x^3 - 7x + 14$, just in a more awkward spot.

Problem 3. We want to compute $\int \sec^8(x)\tan(x)\,dx$. Can you find multiple u that all work?

Solution: The "obvious" u to take is $u = \sec(x)$ so $du = \sec(x)\tan(x)dx$. Then

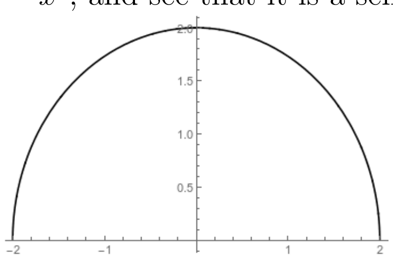
$$\int \sec^8(x)\tan(x)\,dx = \int u^7\,du = \frac{u^8}{8} + C = \frac{\sec^8(x)}{8} + C.$$

It also works to take $u = \sec^2(x)$ or $\sec^4(x)$ or $\sec^8(x)$. e.g. if $u = \sec^8(x)$, then $du = 8\sec^8(x)\tan(x)dx$ and

$$\int \sec^8(x)\tan(x)\,dx = \int \frac{1}{8}\,du = \frac{u}{8} + C = \frac{\sec^8(x)}{8} + C.$$

Problem 4. Evaluate $\int_{-2}^{2} 4\sqrt{4-x^2}\,dx$ by thinking about area. (Hint: what does the graph of $\sqrt{4-x^2}$ look like?)

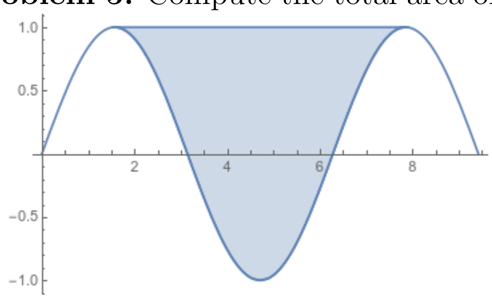
Solution: It's possible to do the integral algebraically, but not really with the techniques we have in this course. (The big idea from Calc 2 is we can do the substitution $x = \sin(u)$ and then use trigonometric identities to simplify the integral.) Instead, we graph the function $\sqrt{4-x^2}$, and see that it is a semicircle with radius 2, and thus has area $\pi(2^2)/2 = 2\pi$.



We multiply by four to get 8π, and thus have

$$\int_{-2}^{2} 4\sqrt{4-x^2}\,dx = 8\pi.$$

Problem 5. Compute the total area of the "valley" between two peaks of the sine function.



Solution: We see that this area is the area of the region between $y = 1$ and $y = \sin x$ between $\pi/2$ and $5\pi/2$. (There are other ways to set this up, but this way works). So we compute

$$\int_{\pi/2}^{5\pi/2} 1 - \sin x \, dx = x + \cos(x)|_{\pi/2}^{5\pi/2} = (5\pi/2 + 0) - (\pi/2 + 0) = 2\pi.$$

Problem 6. We want to find the area of the region bounded by $y = x^2 + 1, y = 17 - x^2$, and $x = 1$, taking the side with $x \geq 1$.

(a) Sketch the region in question. Based on the picture, would you rather integrate with respect to x or to y? Discuss this with someone near you.

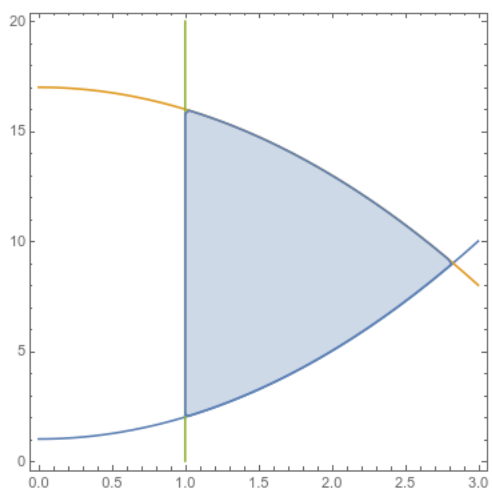
(b) Set up an integral to compute this region, integrating with respect to x.

(c) Set up an integral to compute this region, integrating with respect to y.

(d) Which of these integrals do you prefer? Pick one and compute it.

Solution:

(a) We first draw the region, and see a sort of sideways triangle with a base at $x = 1$ and a point where the curves $y = x^2 + 1$ and $y = 17 - x^2$ intersect. Setting them equal, we get $x^2 + 1 = 17 - x^2$, which gives $2x^2 = 16$ and $x = \pm\sqrt{8}$. Since $x \geq 1$ we know we want $x = \sqrt{8}$, and thus the point of the triangle is at $(\sqrt{8}, 9)$. Checking where the two curves hit $x = 1$ we see that y varies from 1 to 17.



(b) To integrate with respect to x, we see that x varies from 1 to $\sqrt{8}$, and get

$$A = \int_1^{\sqrt{8}} (17 - x^2) - (x^2 - 1) \, dx = \int_1^{\sqrt{8}} 18 - 2x^2 \, dx.$$

(c) To integrate with respect to y, we'd have to write x as a function of y: we see our two curves are $x = \sqrt{y-1}$ and $x = \sqrt{17-y}$. Then we have to break our region into two pieces: one as x goes from 2 to 9, and the other as x goes from 9 to 17.

$$A = \int_2^9 \sqrt{17-y} - 1 \, dy + \int_9^{16} \sqrt{17-y} - 1 \, dy,$$

(d) The second integral is in fact doable, but it's unnecessarily ugly. Instead we integrate with respect to x:

$$A = \int_1^{\sqrt{8}} (17 - x^2) - (x^2 - 1) \, dx = \int_1^{\sqrt{8}} 18 - 2x^2 \, dx$$
$$= 18x - \frac{2}{3}x^3 \Big|_1^{\sqrt{8}} = 36\sqrt{2} - 32\sqrt{2}/3 - 18 + 2/3 = \frac{76\sqrt{2} - 52}{3}.$$