Math 1231 Spring 2023
Single-Variable Calculus 1 Section 12
Optional Mastery Quiz 14
Due Tuesday, May 2

This week's mastery quiz has two topics. Everyone can benefit from submitting M6.

You can submit on Blackboard, or you can hand it in during my office hours on Tuesday. (Or hand it in to the TA directly.)

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 5: Integration
- Major Topic 6: Applications of Integration

Name:

Recitation Section:

Major Topic 5: Integration

(a) Compute $\int x^3\sqrt{x^2+1}\,dx$

Solution: Take $u = x^2+1$ so $du = 2x\,dx$, then $dx = \frac{du}{2x}$ and $x^2 = u-1$. We have

$$\begin{aligned}\int x^3\sqrt{x^2+1}\,dx &= \int x^3\sqrt{u}\frac{du}{2x} = \int \frac{x^2}{2}\sqrt{u}\,du \\ &= \frac{1}{2}\int (u-1)\sqrt{u}\,du = \frac{1}{2}\int u^{3/2}-u^{1/2}\,du \\ &= \frac{1}{2}\left(\frac{u^{5/2}}{5/2}-\frac{u^{3/2}}{3/2}\right)+C \\ &= \frac{u^{5/2}}{5}-\frac{u^{3/2}}{3}+C \\ &= \frac{1}{5}(x^2+1)^{5/2}-\frac{1}{3}(x^2+1)^{3/2}+C.\end{aligned}$$

(b) **By changing the bounds of the integral** compute $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx =$

Solution: Take $u = x^2$ so $du = 2x\,dx$ and $dx = du/2x$. We then have $g(0) = 0^2 = 0$ and $g(\sqrt{\pi}) = \sqrt{\pi}^2 = \pi$ so we get

$$\begin{aligned}\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx &= \int_0^{\pi} x\sin(u)\frac{du}{2x} \\ &= \frac{1}{2}\int_0^{\pi} -\cos(u)\,du = \frac{-1}{2}\cos(u)\Big|_0^{\pi} = \frac{-1}{2}(-1-1) = 1.\end{aligned}$$

(c) Compute $\int x\cos(3x^2-2)\,dx =$

Solution: Set $u = 3x^2-2$ so $du = 6xdx$ and $dx = \frac{du}{6x}$. Then

$$\begin{aligned}\int x\cos(3x^2-2)\,dx &= \int x\cos(u)\frac{du}{6x} = \frac{1}{6}\int\cos(u)\,du \\ &= \frac{1}{6}\sin(u)+C = \frac{1}{6}\sin(3x^2-2)+C.\end{aligned}$$

Major Topic 6: Applications of Integrals

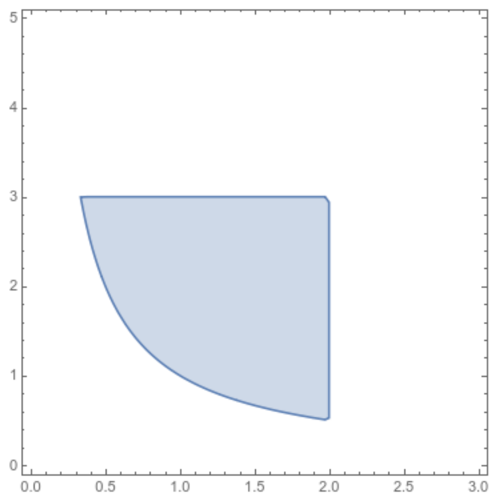
(a) A spring with natural length of 30 cm takes 10 N of force to stretch to 40 cm. How much work does it take to stretch it from 40 cm to 50 cm?

Solution: 40 cm is 10cm beyond the natural length, so we have $10 = k \cdot .1$ and thus $k = 100$. Then the work done is

$$W = \int_{.1}^{.2} 100x\,dx = 50x^2|_{.1}^{.2} = 2 - .5 = 1.5J.$$

(b) Sketch the region bounded by $y = 2x^2, y = 0, x = 0, x = 1$, and find its center of mass.

Solution:



The area of the region is

$$A = \int_0^1 2x^2\,dx = \frac{2}{3}x^3\Big|_0^1 = 2/3.$$

To find the center of mass we compute the moment

$$M_y = \int_0^1 2x^3\,dx = \frac{1}{2}x^4\Big|_0^1 = 1/2.$$

Thus the x-coordinate of the center of mass is

$$\overline{x} = \frac{M_y}{A} = \frac{1/2}{2/3} = 3/4.$$

(c) Find the volume of the solid of revolution formed by rotating the region bounded by $y = x^2$ and $y = x + 2$ about the line $y = 5$.

Solution: We can integrate with respect to x. Our curves intersect when $x^2 = x + 2$ and thus when $0 = x^2 - x - 2 = (x - 2)(x + 1)$, so we integrate from -1 to 2.

Our cross sections are washers. Our outer radius will be $3 - x^2$ and our inner radius will be $5 - (x + 2) = 3 - x$. So our integral is

$$\begin{aligned} V &= \pi \int_{-1}^{2} (5 - x^2)^2 - (3 - x)^2 \, dx \\ &= \pi \int_{-1}^{2} (25 - 10x^2 + x^4) - (9 - 6x + x^2) \, dx \\ &= \pi \int_{-1}^{2} 16 + 6x - 11x^2 + x^4 \, dx \\ &= \pi \left(16x + 3x^2 - \frac{11}{3}x^3 + \frac{1}{5}x^5 \right) \Big|_{-1}^{2} \\ &= \pi \Big(32 + 12 - 88/3 + 32/5 - \big(-16 + 3 + 11/3 - 1/5 \big) \Big) = \frac{63\pi}{5}. \end{aligned}$$