Math 1231: Single-Variable Calculus 1

George Washington University   Spring 2023

Recitation 14

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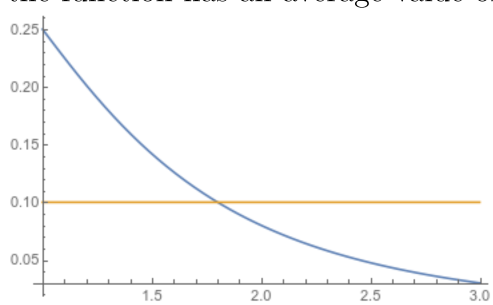
Problem 1. Find the average value of the function $\frac{x}{(x^2+1)^2}$ for $1 \leq x \leq 3$.

Solution: We take $u = x^2 + 1$ so $du = 2x\,dx$, and so

$$\begin{aligned} A &= \frac{1}{3-1}\int_1^3 \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_2^{10} \frac{x}{u^2}\cdot\frac{du}{2x} \\ &= \frac{1}{4}\int_2^{10} u^{-2}\,du = \frac{1}{4}(-u^{-1})\Big|_2^{10} \\ &= \frac{1}{4}\left(\frac{-1}{10} - \frac{-1}{2}\right) = \frac{1}{4}\cdot\frac{2}{5} = \frac{1}{10}. \end{aligned}$$

So the function has an average value of $\frac{1}{10}$ between 1 and 3.



Problem 2. A 12in spring is stretched to 15in by a force of 75lbs.

(a) What is the spring constant? What units does it have?

(b) What is the function that gives force as a function of position? what units does it have?

(c) What is the work done by stretching the spring from 16in to 20in? What units are your answer in?

Solution:

(a) We know that the force must be kx where x is the displacement. Since the displacement is 3in and the force is 75lbs we must have $k = 25\text{lbs/in}$.

(b) We have $F(12) = 0$ and $F(15) = 75$. We can write $F(x) = 25(x - 12)$. This function takes in inches, and outputs force in pounds.

(c) We need to integrate the force over the displacement we're using. So we compute

$$\begin{aligned} W &= \int_{16}^{20} 25(x-12)\,dx = \int_{16}^{20} 25x - 300\,dx \\ &= \frac{25}{2}x^2 - 300x\Big|_{16}^{20} = (5000 - 6000) - (3200 - 4800) \\ &= -1000 - (-1600) = 600. \end{aligned}$$

Thus the work is in lbs. The units are, importantly, foot-inches; in the more usual units of foot-pounds, the work is 50ft lbs.

Problem 3. We want to find the x-coordinate of the center of mass of the semicircle bounded by $y = \sqrt{16 - x^2}$ and $y = 0$, between $x = -4$ and $x = 4$.

(a) Sketch a picture of this region. Geometrically, where should the center of mass be?

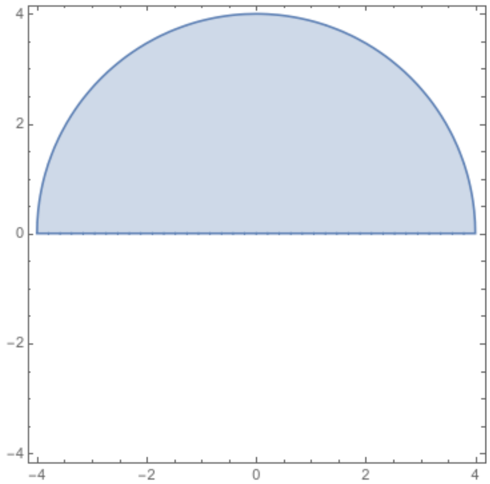
(b) Sketch out the vertical strips you want to cut it into. What is the formula for the area of a strip? (Your formula should involve a dx term.)

(c) Set up an integral to compute the area of the whole region? Can you do this antiderivative? Can you compute the area another way?

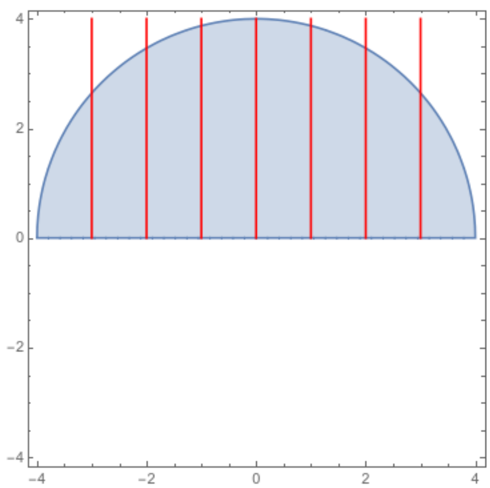
(d) The *moment* of a strip along the x-axis is given by the formula $m_y = xA$. (No, that y is not a typo; it's a moment *along* the x-axis but *around* the y-axis.) What is a formula for the moment of a strip?

(e) Use an integral to calculate the total moment of the region. Then divide by the area to get the center of mass.

Solution:

(a) 

This is horizontally symmetric, so it should probably have a center of mass at $x = 0$.

(b) 

A strip has width dx and height $\sqrt{16 - x^2}$, so the area of a strip is $\sqrt{16 - x^2}\,dx$.

(c) The area is $\int_{-4}^{4} \sqrt{16 - x^2}\,dx$, but we don't have the tools to compute that antiderivative. (We can do it with a calculus 2 technique called trigonometric substitution, which is a clever way to find the admittedly-weird u-substitution $u = \arcsin(x)$; we get the antiderivative $\frac{x}{2}\sqrt{16 - x^2} + 8\arcsin(x/4)$.)

But this is a semicircle of radius 4. The circle has area $\pi \cdot 4^2 = 16\pi$ so this semicircle has area 8π.

(d) The moment of a strip is $x\sqrt{16 - x^2}\,dx$.

(e) The total moment is what we get by adding up the moments of every strip: we get

$$M_y = \int_{-4}^{4} x\sqrt{16 - x^2}\, dx.$$

We can take $u = 16 - x^2$ here. Then $du = -2x\, dx$, and we have $u(-4) = u(4) = 0$. We can take an indefinite integral, and we get

$$\begin{aligned}
\int x\sqrt{16 - x^2}\, dx &= \int x\sqrt{u}\frac{du}{-2x} = \frac{-1}{2}\int \sqrt{u}\, du \\
&= \frac{-1}{2} \cdot \frac{2}{3}u^{3/2} + C = \frac{-1}{3}(16 - x^2)^{3/2} + C. \\
\int_{-4}^{4} x\sqrt{16 - x^2} &= \frac{-1}{3}(16 - x^2)^{3/2}\Big|_{-4}^{4} = \frac{-1}{3}(0)^{3/2} - \frac{-1}{3}(0)^{3/2} = 0.
\end{aligned}$$

Or we could have taken a definite integral. Since $u(-4) = u(4) = 0$ we get

$$\int_{-4}^{4} x\sqrt{16 - x^2}\, dx = \int_{0}^{0} x\sqrt{u}\frac{du}{-2x} = 0$$

since we're integrating from 0 to 0. Thus the total moment is 0, which makes sense because the shape is symmetric.

We can find the center of mass $\overline{x}$ by taking this moment and dividing by 8π. But we still get $\overline{x} = 0$.

Problem 4. Let's find the volume of the solid generated by rotating the region bounded by $y = x$ and $y = \sqrt{x}$ about the line $y = 1$.

(a) Sketch a picture of the region, and draw in the line of revolution.

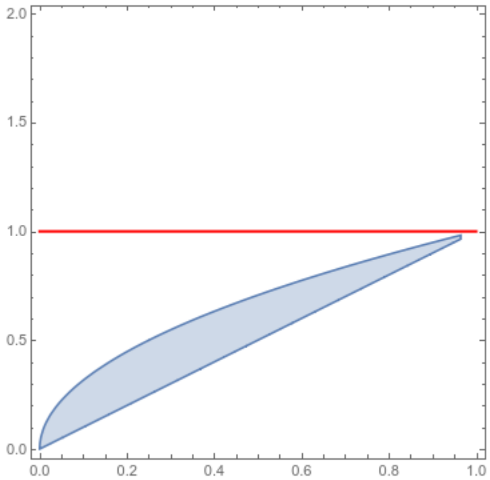
(b) Lightly try to sketch what the solid of revolution will look like. Can you describe it in words?

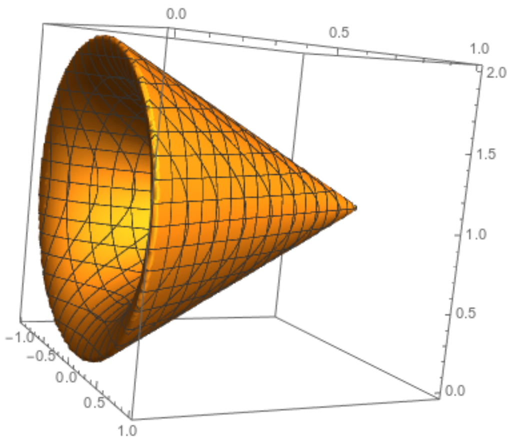
(c) Sketch in the slices you're going to use. Write down a formula for the volume of one slice. (This should involve a dx).

(d) Set up an integral that computes the volume of the whole solid.

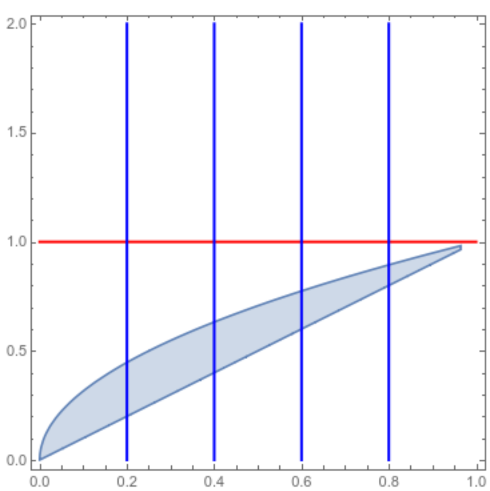
(e) Compute the volume.

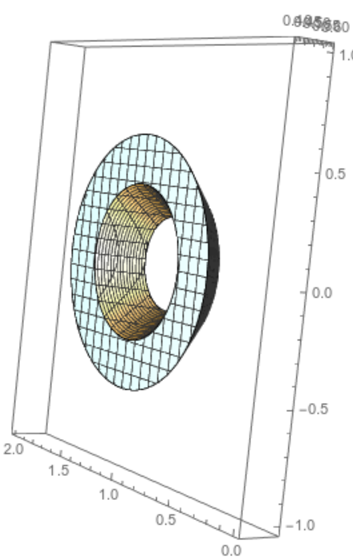
Solution:

(a) 

(b) 

It sort of looks like a cone partially hollowed out on the inside. You can see it as a stack of progressively smaller rings.

(c) 



Each slice is roughly an annulus, with outer radius $1-x$ and inner radius $1-\sqrt{x}$. Thus the volume of a slice is $\pi(1-x)^2\,dx - \pi(1-\sqrt{x})^2\,dx$.

(d)

$$V = \pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2\,dx = \pi \int_0^1 x^2 - 3x + 2\sqrt{x}\,dx.$$

(e)

$$\begin{aligned} V &= \pi \int_0^1 (1-x)^2 - (1-\sqrt{x})^2\,dx = \pi \int_0^1 x^2 - 3x + 2\sqrt{x}\,dx \\ &= \pi \left(\frac{x^3}{3} - \frac{3x^2}{2} + \frac{4}{3}x^{3/2} \right)\Big|_0^1 = \pi \left(\frac{1}{3} - \frac{3}{2} + \frac{4}{3} \right) = \frac{\pi}{6}. \end{aligned}$$