Math 1231 Spring 2023
Single-Variable Calculus 1 Section 12
Mastery Quiz 4
Due Tuesday, February 14

This week's mastery quiz has three topics. Everyone should submit work on S2 and M2. If you already have a 4/4 on M1, which would require you to have gotten full credit both the previous two weeks, you should not submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Major Topic 2: Computing Derivatives
- Secondary Topic 2: Definition of Derivative

Name:

Recitation Section:

Major Topic 1: Computing Limits

Compute the following limits, justifying your answers with clean and correct work.

(a) $\lim_{x\to 3} \frac{x^2-5x+6}{x^2-x-6} =$

Solution:

$$\begin{aligned}\lim_{x\to 3} \frac{x^2-5x+6}{x^2-x-6} &= \lim_{x\to 3} \frac{(x-3)(x-2)}{(x-3)(x+2)} \\ &= \lim_{x\to 3} \frac{x-2}{x+2} = 1/5.\end{aligned}$$

(b) $\lim_{x\to 4^+} \frac{x+1}{x-4} =$

Solution: The limit of the top is 5 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the bottom will always be positive as we approach from the right, the overall limit is in fact $+\infty$.

(c) $\lim_{x\to 0} \frac{x\sin(2x)}{\sin(5x)\sin(3x)} =$

Solution:

$$\lim_{x\to 0} \frac{x\sin(2x)}{\sin(5x)\sin(3x)} = \lim_{x\to 0} \frac{5x}{\sin(5x)} \frac{\sin(2x)}{2x} \frac{3x}{\sin(3x)} \frac{2}{15} = \frac{2}{15}.$$

Major Topic 2: Computing Derivatives

(a) Compute the derivative of $f(x) = (2x^4+3)(5x-2\sqrt{x})$, explicitly justifying each step and naming each derivative rule you use.

Solution:

$$\begin{aligned}f'(x) &= (2x^4+3)'(5x-2\sqrt{x}) + (2x^4+3)(5x-2\sqrt{x})' && \text{Product Rule} \\ &= \Big((2x^4)' + (3)'\Big)(5x-2\sqrt{x}) + (2x^4+3)\Big((5x)' - (2\sqrt{x})'\Big) && \text{Sum Rule} \\ &= \Big((2x^4) + 0\Big)(5x-2\sqrt{x}) + (2x^4+3)\Big((5x)' - (2\sqrt{x})'\Big) && \text{Constants Rule} \\ &= \Big(2(x^4)'\Big)(5x-2\sqrt{x}) + (2x^4+3)\Big(5(x)' - 2(\sqrt{x})'\Big) && \text{Scalar Products} \\ &= \Big(2(x^4)'\Big)(5x-2\sqrt{x}) + (2x^4+3)\Big(5\cdot 1 - 2(\sqrt{x})'\Big) && \text{Identity} \\ &= (2\cdot 4x^3)(5x-2\sqrt{x}) + (2x^4+3)\left(5\cdot 1 - 2\cdot\frac{1}{2}x^{-1/2}\right) && \text{Power Rule.}\end{aligned}$$

(b) Compute the derivative of $g(x) = \frac{5x^4 - 3x^2}{x^5 + \sqrt[5]{x} + 7}$.

Solution:

$$g'(x) = \frac{(20x^3 - 6x)(x^5 + \sqrt[5]{x} + 7) - (5x^4 + \frac{1}{5}x^{-4/5})(5x^4 - 3x^2)}{(x^5 + x + 7)^2}.$$

(c) Compute the derivative of $h(x) = \frac{5}{x^4}$.

Solution: There are two ways you could approach this but you need to do it correctly consistently; this is question that trips people up very frequently.

One option is to use the quotient rule. But if we do that we have to remember that $5' = 0$. So we get

$$h'(x) = \frac{(5)'x^4 - 5(x^4)'}{(x^4)^2} = \frac{0 - 20x^3}{x^8} = \frac{-20}{x^5}.$$

The other option is to rewrite this as an exponent. Then we have

$$h'(x) = \frac{d}{dx}5x^{-4} = 5 \cdot (-4x^{-5}) = -20x^{-5}.$$

I think the second approach is better and more consistent. But you *can* use the quotient rule as long as you do it correctly.

Secondary Topic 2 Definition of Derivative

Compute the following derivatives, *directly from the formal definition of derivative.*

(a) If $f(x) = x^2 + 2x$, find $f'(2)$.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \lim_{h \to 0} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 6h}{h} \\ &= \lim_{h \to 0} h + 6 = 6. \end{aligned}$$

(b) If $g(x) = \frac{1}{x+2}$, find $g'(x)$.

Solution:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\ &= \lim_{h\to 0} \frac{(x+2)-(x+h+2)}{(x+2)(x+h+2)h} \\ &= \lim_{h\to 0} \frac{-h}{(x+2)(x+h+2)h} \\ &= \lim_{h\to 0} \frac{-1}{(x+2)(x+h+2)} = \frac{-1}{(x+2)^2}. \end{aligned}$$