Math 1231: Single-Variable Calculus 1
George Washington University Spring 2023
Recitation 6

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February 24, 2023

Problem 1 (Geometric Series)**.** Another function it's sometimes important to approximate is the "geometric series" formula $f(x) = \dfrac{1}{1-x}$, near $x = 0$.

(a) What is $f'(x)$?

(b) Find a linear approximation for $f(x)$ near $x = 0$.

(c) Use this formula to estimate $\frac{1}{.9}$ and $\frac{1}{1.01}$. Do these answers make sense?

(d) Use your formula to estimate $\frac{1}{1.5}$ and $frac10.5$. Do these answers make sense?

(e) Use your formula to estimate $f(-1)$ and $f(1)$. Do these answers make sense?

Solution:

(a) $f'(x) = -(1-x)^{-2} = \frac{1}{(1-x)^2}$. This is tricky; you get a negative sign from the power rule, but another from the chain rule that cancels it out.

(This is a weird way to write the function! Why not just use $\frac{1}{1+x}$? Because this setup makes more sense in a lot of the applications people want to use it for. You'll see why when you study power series in Calculus 2.)

(b) $f'(0) = 1$, so our linear approximation is $f(x) \approx 1 + x$.

(c) $\frac{1}{.9} = f(.1) \approx 1.1$. The true answer is $1.\overline{11}$, so that checks out.

$\frac{1}{1.01} = f(-.01) \approx .99$. The true answer is $.990099\ldots$, which also makes sense.

(d) $\frac{1}{1.5} = f(-0.5) \approx 0.5$. The true answer is $2/3 \approx .\overline{66}$ so this is, like, okay-ish.

$\frac{1}{0.5} = f(0.5) \approx 1.5$. The true answer is 2, so this is again okay, but not great.

(e) $f(-1) \approx 0$. But $f(-1) = 1/2$, so that doesn't make a ton of sense. This is because (-1) is "far away" from zero for our purposes. And how do we know it's far away? Well...

$f(1) \approx 0$. But $f(1)$ is utterly undefined, since it asks us to divide by 0. We've gone too far away for the linear approximation to work at all.

Now let's look at how we can *use* some of the formulas we've come up with.

Problem 2. (a) Let $g(x) = \frac{1}{1-x^2}$. We can linearly approximate this near $x = 0$; what do we get?

(b) But we can do this a different way. If we keep $f(x) = \frac{1}{1-x}$ the geometric series formula, we can view this function as $f(x^2)$. What does our approximation from problem **??** tell us that $f(x^2)$ is?

(c) Which of these will work better? Why? What makes (b) different from (a)?

Solution:

(a) $g'(x) = -(1-x^2)^{-2} \cdot (-2x) = \frac{2x}{(1-x^2)^2}$. So $g'(0) = 0$, and our linear approximation is $g(x) \approx 1 + 0(x-0) = 1$.

(b) $f(x) \approx 1 + x$ so $f(x^2) \approx 1 + x^2$.

(c) (b) is going to be better, because it takes more information into account. (a) gives the best linear approximation, but (b) gives a *better* approximation, because it's *quadratic*.

Problem 3. We can do the same thing with another function. Let's take $h(x) = \frac{1}{1+x}$.

(a) We can do a straightforward linear approximation of h near 0. What do we get?

(b) We can also use our geometric series formula $f(x) = \frac{1}{1-x} \approx 1 - x$ here. Can we rewrite $h(x)$ in terms of f, like we did for g above? What does that give us as an approximation?

(c) There's one more way we can approximate this function. Remember the *binomial approximation* formula $(1+x)^\alpha \approx 1 + \alpha x$. How does that apply here, and what do we get?

Solution:

(a) $h'(x) = \frac{-1}{(1+x)^2}$ so $h'(x) = -1$, and we get $h(x) \approx 1 - x$.

(b) Here $h(x) = f(-x)$. So we have $f(-x) \approx 1 + (-x) = 1 - x$, the same as in part (a).

(c) $g(x) = (1 + x)^{-1}$ so we take $\alpha = -1$, and the binomial formula gives us $g(x) \approx 1 + (-1)x = 1 - x$, which is the same answer again.

It maybe shouldn't surprise us that we get the same answer three ways: each of these is a linear approximation to the same function, so they should all be the same.

Problem 4. (a) Use the binomial approximation to estimate $\sqrt{2}$ and $\sqrt[n]{2}$.

(b) Use the binomial approximation to estimate $\sqrt{17}$. (Remember: 17 is not close to 1! You need to be slightly clever here.)

(c) Can you find a formula to approximate $(1 + x^n)^\alpha$ for a real number α?

(d) What does this tell us about $\sqrt{1 + x^2}$?

Solution:

(a) We have $\sqrt{2} = (1 + 1)^{1/2} \approx 1 + \frac{1}{2} \cdot 1 \approx 3/2$, and $\sqrt[n]{2} = (1 + 1)^{1/n} \approx 1 + \frac{1}{n} = \frac{n+1}{n}$.

(b)

$$\begin{aligned}\sqrt{17} &= \sqrt{16 \cdot 17/16} = 4\sqrt{17/16} = 4\sqrt{1 + 1/16} \\ &= 4(1 + 1/16)^{1/2} \approx 4\left(1 + \frac{1}{32}\right) \\ &= 4 + \frac{1}{8} = 4.125.\end{aligned}$$

The true answer is $4.12311\ldots$.

(c) By the binomial approximation, $(1 + x^n)^\alpha \approx 1 + \alpha x^n$.

(d) Thus in particular, $\sqrt{1 + x^2} \approx 1 + \frac{1}{2}x^2$. Note that this is very different from $1 + x$!

Problem 5 (Bonus)**.** Find a formula to approximate $f(x) = x^3 + 3x^2 + 5x + 1$ near $a = 0$. What do you notice? Why does that happen?

Solution: We have $f(0) = 1$ and $f'(x) = 3x^2 + 6x + 5$ so $f'(0) = 5$. Thus

$$f(x) \approx 1 + 5x.$$

This is exactly what you get if you take the original polynomial and cut off all the terms of degree higher than 1.

This makes sense, because we're looking for the closest we can get to f without using terms of degree higher than 1.

Problem 6. Suppose a particle has height as a function of time given by $h(t\text{s}) = (2t^3 - 3t^2 - 12t + 3)\,\text{m}$.

(a) What is the velocity of this particle at time $t = 0$? What are the units, and why?

(b) What is the acceleration of this particle at time $t = 0$? What are the units and why?

(c) When is the particle speeding up? When is it slowing down?

Solution:

(a) $h'(t) = (6t^2 - 6t - 12)\,\text{m/s}$ so $h'(0) = -12\text{m/s}$. We get m/s because the input is in seconds and the output is in meters, so the derivative, which is $\frac{\Delta h}{\delta t}$, is meters over seconds.

(b) $h''(t) = (12t - 6)\,\text{m/s}^2$ so $h''(0) = 12\text{m/s}^2$. Here, the derivative is $\frac{\Delta h'}{\Delta t}$, so the numerator is in m/s and the denominator is in s, giving m/s^2.

(c) The particle is speeding up when the derivative is increasing. This would have to mean the second derivative is positive, and thus we want $12t > 6$ and so $t > 1/2$. The particle is slowing down when $t < 1/2$.

Problem 7. Suppose we have a function $C(t)$ that tells us the concentration of a drug in the bloodstream as a function of time. Specifically, if we give t as the number of hours since the drug has been taken, C is the concentration in milligrams per liter.

(a) Write down the formula for the *definition of the derivative* of C, at time t_0. What are the units in this formula?

(b) What are the units of the derivative $C'(t)$?

(c) What does the derivative tell us, and why?

Solution:

(a) We have

$$C'(t_0) = \lim_{t \to t_0} \frac{C(t)\mathrm{mg/L} - C(t_0)\mathrm{mg/L}}{t\mathrm{h} - t_0\mathrm{h}}$$
$$= \lim_{h \to 0} \frac{C(t_0 + h)\mathrm{mg/L} - C(t_0)\mathrm{mg/L}}{h\mathrm{h}}.$$

The numerator has units milligrams per liter, and the denominator has units of hours.

(b) Therefore the derivative has units milligrams per liter per hour, or milligrams per liter-hour.

(c) The derivative is measuring the change in concentration divided by the change in time; so it's the change in concentration per hour. It tells us how quickly the blood concentration is going up or down over time. Thus the units are concentration per time, or (milligrams per liter) per hour.