Math 1231 Spring 2023
Single-Variable Calculus 1 Section 12
Mastery Quiz 6
Due Tuesday, February 28

This week's mastery quiz has four topics. Everyone should submit work on M3 and S4. If you already have a 4/4 on M2, you should not submit it. But if Blackboard doesn't say you're at a 4/4, then you should submit again for another try. If you already have a 2/2 on Rates of Change, you shouldn't submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Major Topic 3: Linear Approximation
- Secondary Topic 3: Rates of Change
- Secondary Topic 4: Related Rates

Name:

Recitation Section:

Major Topic 2: Computing Derivatives

(a) Find a formula for y' in terms of x and y if $x^3y + x^2y^2 + y^4 = 0$.

Solution:

$$\begin{aligned}
3x^2y + x^3y' + 2xy^2 + 2x^2yy' + 4y^3y' &= 0\\
x^3y' + 2x^2yy' + 4y^3y' &= -3x^2y - 2xy^2\\
y' &= -\frac{3x^2y + 2xy^2}{x^3 + 2x^2y + 4y^3}.
\end{aligned}$$

(b) Compute $\frac{d}{dx}g(x) = \left(\frac{x\csc(x)}{\sqrt{x^3 - x}}\right)^3$

Solution:

$$g'(x) = 3\left(\frac{x\csc(x)}{\sqrt{x^3 - x}}\right)^2 \frac{\big(\csc(x) - x\csc(x)\cot(x)\big)\sqrt{x^3 - 1} - x\csc(x)\frac{1}{2}(x^3 - x)^{-1/2}(3x^2 - 1)}{x^3 - x}.$$

Major Topic 3: Linear Approximation

(a) Write a tangent line to the curve $x^2y^2 = 5 + x + y$ at the point $(1, 3)$.

Solution: Implicit differentiation gives us

$$\begin{aligned}
2xy^2 + 2x^2yy' &= 1 + y'\\
2\cdot 1\cdot 9 + 2\cdot 1^2\cdot 3\cdot y' &= 1 + y'\\
18 + 6y' &= 1 + y'\\
5y' &= -17\\
y' &= -17/5
\end{aligned}$$

and thus the tangent line has equation

$$y - 3 = \frac{-17}{5}(x - 1).$$

Alternatively we can compute

$$\begin{aligned}
2xy^2 + 2x^2yy' &= 1 + y'\\
y'(2x^2y - 1) &= 1 - 2xy^2\\
y' &= \frac{1 - 2xy^2}{2x^2y - 1}
\end{aligned}$$

(b) Use linear approximation to estimate $\sqrt[4]{14}$.

Solution: We take $f(x) = \sqrt[4]{x}$, and take $a = 16$. Then

$$f'(x) = \frac{1}{4}x^{-3/4}$$
$$f'(16) = \frac{1}{4}(16)^{-3/4} = \frac{1}{4 \cdot 8} = \frac{1}{32}$$
$$f(x) \approx f(a) + f'(a)(x-a) = 2 + \frac{1}{32}(14-16) = 2 - \frac{1}{16} = \frac{31}{16}.$$

Secondary Topic 3: Rates of Change

(a) The force a magnet exerts on a piece of iron depends on the distance between the magnet and the metal. Let $F(d) = \frac{2}{d^2}$ give the force exerted by the magnet in Newtons, where d is the distance between them in meters.

(i) What are the units of $F'(d)$? What does it $F'(d)$ represent physically? What would it mean if $F'(d)$ is big?

Solution: The derivative is the rate at which the amount of force changes as you change the distance between the magnet and the iron; its units are Netwons per meter. If $F'(d)$ is big, that means that moving the magnet a little bit will change the force on it by a lot.

(ii) Calculate $F'(2)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution: $F'(d) = \frac{-4}{d^3}$ so $F'(3) = \frac{-4}{8} = -1/2$. This means that moving the iron another meter away from the magnet should reduce the force by about half a Newton.

(b) Suppose the distance between two particles in centimeters is given as a function of time in seconds by the formula $d(t) = t + \frac{1}{t}$.

(i) When is the velocity zero?

Solution: $d'(t) = 1 - 1/t^2$ so the velocity is zero when $t = \pm 1$.

(ii) When is the acceleration zero?

Solution: $d''(t) = 2/t^3$ is never zero.

Secondary Topic 4: Related Rates

A rocket is taking off with a perfectly vertical path, and is being tracked by a radar station on the ground four miles from the launch pad. We want to know how fast the rocket is rising when it is three miles high and its distance from the radar station is increasing at a rate of 3000 miles per hour.

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence.

Solution: We know one speed and want to know another, and we also know distances. This means we probably want to use the distance formula and take its derivative to find speeds.

We write $h = 3\text{mi}$, and can work out that $d = 5\text{mi}$. We know from the text of the problem that $d' = \text{mi/h}$.

We know that $d^2 = h^2 + 4^2\text{mi}^2$ and thus $2dd' = 2hh'$. Plugging in values gives us

$$\begin{aligned} 2 \cdot 5\text{mi} \cdot 3000\text{mi/hr} &= 2 \cdot 3\text{mi} \cdot h' \\ h' &= 5000\text{mi/hr}. \end{aligned}$$

Thus the rocket is rising at 5000 miles per hour.