Math 1231: Single-Variable Calculus 1
George Washington University Spring 2023
Recitation 7

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Problem 1. Find the tangent line to $y = 6\cos x$ at $(\pi/3, 3)$.

Solution: We see that $y' = -6\sin x$, and thus when $x = \pi/3$ we have $y' = -3\sqrt{3}$. Recalling that the equation of our line is $y = m(x - x_0) + f(x_0)$, we have the equation $y = -3\sqrt{3}(x - \pi/3) + 3$.

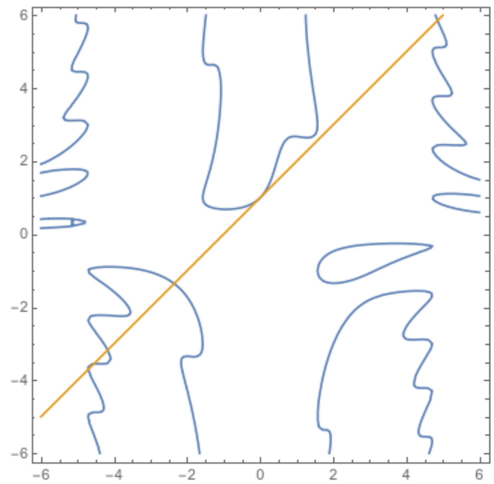
Problem 2. Find an equation for a tangent line to $y\cos(x) = 1 + \sin(xy)$ at the point $(0, 1)$.

Solution: We can compute

$$\frac{d}{dx}\left(y\cos(x)\right) = \frac{d}{dx}\left(1 + \sin(xy)\right)$$
$$\frac{dy}{dx}\cos(x) - y\sin(x) = \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

At this point we could simplify this expression, but we *shouldn't*. Instead, we can plug in $(0, 1)$ now, and get

$$\frac{dy}{dx}\cos(0) - 1\sin(0) = \cos(0)\left(1 + 0\frac{dy}{dx}\right)$$
$$\frac{dy}{dx} = 1$$
$$y - 1 = 1(x - 0)$$
$$y = x + 1.$$



Now, we *could* find a formula for y' in terms of x and y; that would give something like this.

$$\frac{dy}{dx}\left(\cos(x) - x\cos(xy)\right) = y\cos(xy) + y\sin(x)$$
$$\frac{dy}{dx} = \frac{y\cos(xy) + y\sin(x)}{\cos(x) - x\cos(xy)}.$$

But we don't need to for the question that we asked.

Problem 3 (Bonus)**.** (a) If $\sqrt{xy} = x^2y - 2$, find a formula for $\frac{dy}{dx}$ in terms of x and y.

(b) Find an equation of the tangent line at the point $(1, 4)$.

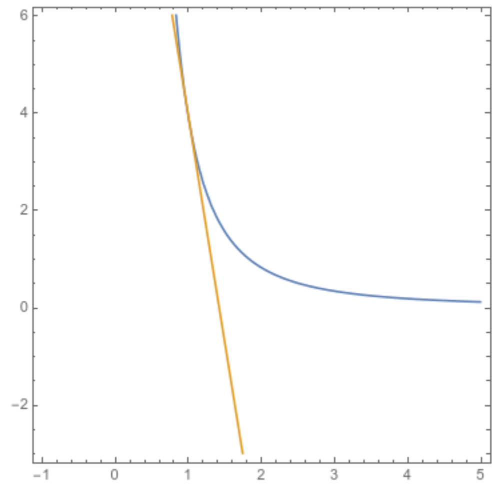
Solution:

(a)

$$\frac{d}{dx}\sqrt{xy} = \frac{d}{dx}\left(x^2y - 2\right)$$
$$\frac{1}{2}(xy)^{-1/2}\left(y + x\frac{dy}{dx}\right) = 2xy + x^2\frac{dy}{dx}$$
$$\frac{dy}{dx}\left(x^2 - \frac{1}{2}x(xy)^{-1/2}\right) = \frac{1}{2}(xy)^{-1/2}y - 2xy$$
$$\frac{dy}{dx} = \frac{\frac{1}{2}(xy)^{-1/2}y - 2xy}{x^2 - \frac{1}{2}x(xy)^{-1/2}}.$$

(b)

$$\frac{dy}{dx}(1,4) = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot 4 - 8}{1 - \frac{1}{2}\cdot\frac{1}{2}} = \frac{-7}{3/4} = \frac{-28}{3}$$
$$y - 4 = \frac{-28}{3}(x - 1).$$



Problem 4. A twenty foot ladder rests against a wall. The bit on the wall is sliding down at 1 foot per second. How quickly is the bottom end moving when the top is 12 feet from the ground?

(a) Draw a picture of this situation.

(b) What is the question you're trying to answer? What do you expect it to look like? Should it be positive or negative? What units do you expect?

(c) What equation should we use here, and why?

(d) Use a derivative to calculate the answer to the question. Does your answer make sense?

Solution: Let h be the height of the ladder on the wall, and b be the distance of the foot of the ladder from the wall. Then $h = 12, h' = -1$, and $b = \sqrt{400 - 144} = 16$. We have

$$h^2 + b^2 = 400$$
$$2hh' + 2bb' = 0$$
$$2 \cdot 12 \cdot (-1) + 2 \cdot 16 \cdot b' = 0$$
$$b' = \frac{24}{32} = 3/4$$

so the foot of the ladder is sliding away from the wall at 3/4 ft/s. Again, the direction of the sliding is correct (away from the wall), and the number seems plausible.

Problem 5. A rectangle is getting longer by one inch per second and wider by two inches per second. When the rectangle is 5 inches long and 7 inches wide, how quickly is the area increasing?

(a) Draw a picture of this situation.

(b) What is the question you're trying to answer? What do you expect it to look like? Should it be positive or negative? What units do you expect?

(c) What equation should we use here, and why?

(d) Use a derivative to calculate the answer to the question. Does your answer make sense?

(e) To check things: how long and wide will the rectangle be after one inch? How much will the area have increased? Does that make sense with your answer to the related rates problem?

(f) Bonus: where have we seen basically this argument before?

Solution:

(a) It's a rectangle.

(b) We want to know how quickly the area is increasing, so we're looking for $\frac{dA}{dt}$, and the units should be in^2/s.

(c) We can relate all our quantities with the formula for the area of a rectangle: $A = \ell w$ relates the area, which we want to know about, to the length and width, which we do know about.

We have $\ell = 5\text{in}, w = 7\text{in}, \frac{d\ell}{dt} = 1\text{in/s}, \frac{dw}{dt} = 2\text{in/s}$. Taking a derivative gives us

$$\begin{aligned}\frac{dA}{dt} &= \ell\frac{dw}{dt} + w\frac{d\ell}{dt}\\ &= 5\text{in}\cdot 2\text{in/s} + 7\text{in}\cdot 1\text{in/s}\\ &= 17\text{in}^2/\text{s}.\end{aligned}$$

The units are right (the rate at which area is changing per second), and the direction is right (the area should be increasing, and this derivative is positive). It's really hard to see if the size is right using our intuition; people in general have bad intuition for the rate at which area changes in response to lengths.

One second later, we'd have $\ell = 6\text{in}$ and $w = 9\text{in}$ for a total area of 54in^2. This is an increase of 19in^2 over our starting area of 35in^2, and 17 is a pretty good approximation of 19.

The derivative of the area formula is just the product rule; we saw basically this same picture during the proof of the product rule.

Problem 6 (Bonus)**.** A kite is flying 100 feet over the ground, moving horizontally at 8 ft/s. At what rate is the angle between the string and the ground decreasing when 200ft of string is let out?

(a) Draw a picture of this situation.

(b) What is the question you're trying to answer? What do you expect it to look like? Should it be positive or negative? What units do you expect?

(c) Of the numbers in the problem and in your picture, which are variables and which are constants? This is an important question, and sometimes a tricky one.

(d) What equation should we use here, and why? Note: this uses a relationship we haven't used yet in class. Look at the picture carefully and think about what quantities you need to relate!

(e) Use a derivative to calculate the answer to the question. Does your answer make sense?

Solution: Call the distance between the kite-holder and the kite d and the angle between the string and the ground θ. When the length of string is 200 then $d = \sqrt{200^2 - 100^2} = 100\sqrt{3}$. We have that $d' = 8$ (since the angle is decreasing, the kite must be getting farther away). And finally we have the relationship $\tan\theta = \frac{100}{d}$ by the definition of tan in terms of triangles. Then we have

$$\begin{aligned} \tan\theta &= 100d^{-1} \\ \sec^2(\theta)\theta' &= -100d^{-2}d' \\ \theta' &= \frac{-100 \cdot 8\cos^2(\theta)}{d^2}. \end{aligned}$$

We see that $\cos(\theta) = \frac{100\sqrt{3}}{200} = \sqrt{3}2$, so we have

$$\theta' = \frac{-100 \cdot 8 \cdot 3/4}{(100\sqrt{3})^2} = -\frac{8}{100 \cdot 4} = \frac{-1}{50}.$$

So the angle between the string and the ground is decreasing at a rate of $1/50$ per second. (Note: radians are unitless, so these really are the units!)