# Math 1231 Spring 2023
# Single-Variable Calculus 1 Section 12
# Mastery Quiz 8
# Due Tuesday, March 21

This week's mastery quiz has three topics. Everyone should submit work on M4 and S4. If you already have a 4/4 on M3, you should not submit it. But if Blackboard doesn't say you're at a 4/4, then you should submit again for another try. If you already have a 2/2 on S4, you shouldn't submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

**Topics on This Quiz**

- Major Topic 3: Linear Approximation
- Major Topic 4: Optimization
- Secondary Topic 4: Related Rates

## Name:

## Recitation Section:

# Major Topic 3: Linear Approximation

(a) Find an equation for the line tangent to the curve $x^2y - xy^3 = xy + 3$ at the point $(3, 1)$.

**Solution:**

$$\begin{aligned} 2xy + x^2y' - y^3 - 3xy^2y' &= y + xy' \\ 6 + 9y' - 1 - 9y' &= 1 + 3y' \\ 4 &= 3y' \\ y' &= 4/3 \end{aligned}$$

and thus an equation for the tangent line is

$$y - 1 = \frac{4}{3}(x - 3).$$

(b) Find a linear approximation to the function $f(x) = \frac{x^3}{1+x}$ near the point $a = 1$ and use it to approximate $f(1.3)$.

**Solution:**

$$\begin{aligned} f(1) &= \frac{1}{2} \\ f'(x) &= \frac{3x^2(1+x) - x^3}{(1+x)^2} \\ f'(1) &= \frac{6-1}{4} = \frac{5}{4} \\ f(x) &\approx \frac{1}{2} + \frac{5}{4}(x-1) \\ f(1.3) &\approx \frac{1}{2} + \frac{5}{4} \cdot .3 = \frac{20}{40} + \frac{15}{40} = \frac{35}{40} = \frac{7}{8}. \end{aligned}$$

# M4: Extrema and Optimization

(a) Find the absolute extrema of $g(x) = 3x^4 - 2x^3 - 3x^2 + 5$ on the interval $[-1, 2]$, and justify your claim that these are the absolute extrema.

**Solution:** $g$ is continuous on the closed interval $[-1, 2]$, so by the Extreme Value Theorem it has a maximum and a minimum on the interval. This must happen at a critical point or an endpoint. (This argument is necessary! Otherwise there's no reason to expect the largest local max to be a global max.)

We compute

$$g'(x) = 12x^3 - 6x^2 - 6x = 6x(2x^2 - x - 1) = 6x(2x+1)(x-1)$$

which is zero at $0, 1, -1/2$. Then we compute

$$\begin{aligned} g(-1) &= 7 \\ g(-1/2) &= \frac{3}{16} + \frac{1}{4} - \frac{3}{4} + 5 = 5 - \frac{5}{16} = \frac{75}{16} = 4.6875 \\ g(0) &= 5 \\ g(1) &= 3 \\ g(2) &= 48 - 16 - 12 + 5 = 25. \end{aligned}$$

Thus $g$ has an absolute maximum of 25 at 2, and an absolute minimum of 3 at 1.

(b) Find all the critical points of $f(x) = \sqrt[3]{x^3 - 3x}$.

**Solution:** We have

$$f'(x) = \frac{1}{3}(x^3 - 3x)^{-2/3}(3x^2 - 3) = \frac{(x-1)(x+1)}{\sqrt[3]{x(x^2-3)}^2}.$$

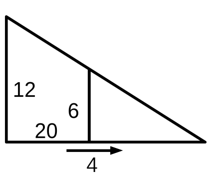
This is zero when $x = \pm 1$ and is undefined when $x = 0$ or $x = \pm\sqrt{3}$. Thus the critical points are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$.

# Secondary Topic 4: Related Rates

A street light is mounted at the top of a 12-foot-tall pole. A six-foot-tall man walks straight away from the pole at 4 feet per second. We want to know how fast the length of his shadow is changing when he is twenty feet from the pole.

(a) Choose an equation to use for this problem, and explain why you chose that equation.

(b) Use calculus to answer the question. Make sure you answer with a complete sentence.




**Solution:** After drawing a picture, we see we have two triangles in the same shape: we know how one triangle is changing, and we want to figure out how the other is changing, so we should relate those similar triangles.

Let $d$ be the distance of the man from the pole. Then $d = 20$ and $d' = 4$. If $s$ is the length of the shadow, then we have $s/6 = (d+s)/12$ so we get

$$\begin{aligned} s &= \frac{d+s}{2} \\ s' &= d'/2 + s'/2 \\ s'/2 &= d'/2 \\ s' &= d' = 4. \end{aligned}$$

Thus the length of the shadow is growing at 4 feet per second.