# Math 1231 Spring 2023
# Single-Variable Calculus 1 Section 12
# Mastery Quiz 9
# Due Tuesday, March 28

This week's mastery quiz has three topics. Everyone should submit work on M4 and S5. If you already have a 4/4 on M3, you should not submit it. But if Blackboard doesn't say you're at a 4/4, then you should submit again for another try.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

**Topics on This Quiz**

- Major Topic 3: Linear Approximation
- Major Topic 4: Optimization
- Secondary Topic 5: Curve Sketching

## Name:

## Recitation Section:

# Major Topic 3: Linear Approximation

(a) Find an equation for the line tangent to the curve $3x^2y + 5xy^2 = 2x$ at the point $(1, -1)$.

**Solution:**

$$\begin{aligned} 6xy + 3x^2y' + 5y^2 + 10xyy' &= 2 \\ -6 + 3y' + 5 - 10y' &= 2 \\ -7y' &= 3 \\ y' &= -3/7 \end{aligned}$$

and thus an equation for the tangent line is

$$y + 1 = \frac{-3}{7}(x - 1).$$

(b) Find a linear approximation to the function $f(x) = \sin(x)\cos(x)$ near the point $a = \pi/3$ and use it to approximate $f(\pi/2)$.

**Solution:**

$$\begin{aligned} f(\pi/3) &= \frac{\sqrt{3}}{4} \\ f'(x) &= \cos^2(x) - \sin^2(x) \\ f'(\pi/3) &= 1/4 - 3/4 = -1/2 f(x) \qquad \approx \frac{\sqrt{3}}{4} - \frac{1}{2}(x - \pi/3) \\ f(\pi/2) &\approx \frac{\sqrt{3}}{4} - \frac{1}{2}(\pi/2 - \pi/3) = \frac{\sqrt{3}}{4} - \frac{\pi}{12}. \end{aligned}$$

# M4: Extrema and Optimization

(a) Classify the critical points and relative extrema of $g(x) = \dfrac{2x - 1}{x^2 + 2}$.

**Solution:** We have

$$\begin{aligned} g'(x) &= \frac{2(x^2 + 2) - 2x(2x - 1)}{(x^2 + 2)^2} = \frac{-2x^2 + 2x + 4}{(x^2 + 2)^2} \\ &= -2\frac{x^2 - x - 2}{(x^2 + 2)^2} = -2\frac{(x - 2)(x + 1)}{(x^2 + 2)^2} \end{aligned}$$

so the critical points are 2 and $-1$. (The derivative is defined everywhere).

To classify these critical points we need to use either the first or second derivative test. I think the first derivative test looks easier here, purely because I don't want to compute the second derivative. I get the table

| | $x-2$ | $x+1$ | $\frac{-2}{(x^2+2)^2}$ | $g'(x)$ |
|---|---|---|---|---|
| $x<-1$ | $-$ | $-$ | $-$ | $-$ |
| $-1<x<2$ | $-$ | $+$ | $-$ | $+$ |
| $2<x$ | $+$ | $+$ | $-$ | $-$ |

Thus we see that there is a relative minimum at $-1$ and a relative maximum at 2.

But we could use the second derivative test if we really wanted to. We compute

$$g''(x) = -2\frac{(2x-1)(x^2+2)^2 - 2(x^2+2)2x(x^2-x-2)}{(x^2+2)^4}$$

$$g''(-1) = -2\frac{(-3)(3)^2 - 2(3)(-2)(0)}{3^4} = \frac{-2\cdot(-27)}{3^4} = 2/3 > 0$$

$$g''(2) = -2\frac{3(6)^2 - 2(6)4(0)}{6^4} = \frac{-1}{6} < 0.$$

Thus $g''(-1) > 0$ so $g$ has a minimum at $-1$; and $g''(2) < 0$ so $g$ has a maximum at 2.

(b) Find the absolute extrema of $g(x) = x^3 - 3x^2 - 9x + 3$ on $[-2, 4]$, and justify your claim that these are in fact absolute extrema.

**Solution:** $g$ is continuous on the closed interval $[-2, 4]$ so by the extreme value theorem it has an absolute maximum and an absolute minimum.

We compute $g'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$ is always defined, and is zero if $x = -1$ or $x = 3$. So the critical points are $-1$ and 3, and we need to check the points $-2, -1, 3, 4$.

$$g(-2) = 1 \qquad g(-1) = 8$$
$$g(3) = -24 \qquad g(4) = -17.$$

Thus $g$ has a maximum of 8 at $-1$ and a minimum of $-24$ at 3.

# S5: Curve Sketching

Let $f(x) = \frac{(x-2)^2}{x-1}$. We can compute that

$$f'(x) = \frac{x(x-2)}{(x-1)^2}$$

$$f''(x) = \frac{2}{(x-1)^3}.$$

Sketch a graph of $f$. Your answer should discuss the domain, roots, asymptotes, limits at infinity, critical points and values, intervals of increase and decrease, and concavity and points of inflection.

**Solution:** The function is defined for all real numbers except 1. We compute that $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = -\infty$. There is a root of $f$ at $x = 2$, and we compute that $\lim_{x\to +\infty} f(x) = +\infty$ and $\lim_{x\to -\infty} f(x) = -\infty$.

$f'(x)$ is undefined at $x = 1$ and is 0 at $0, 2$ so the critical points are $0, 1, 2$. We compute that $f(0) = -4$ and $f(2) = 0$. We make a chart:

| | $x$ | $x-2$ | $(x-1)^{-2}$ | $f'(x)$ |
|---|---|---|---|---|
| $x < 0$ | $-$ | $-$ | $+$ | $+$ |
| $0 < x < 1$ | $+$ | $-$ | $+$ | $-$ |
| $1 < x < 2$ | $+$ | $-$ | $+$ | $-$ |
| $2 < x$ | $+$ | $+$ | $+$ | $+$ |

Thus $f$ is increasing on $(-\infty, 0)$ and $(2, +\infty)$, and is decreasing on $(0, 2)$. It has relative a relative maximum at $(0, -4)$ and a relative minimum at $(2, 0)$; it doens't have a value at 1.

$f''(x)$ is undefined at 1 and is never 0, so the only possible point of inflection is 1. We see that $f''(x)$ is negative if $x < 1$ and positive if $x > 1$, so $f$ is concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$.