Math 1232 Midterm Solutions

Instructor: Jay Daigle

- You will have 75 minutes for this test.
- You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may not use a calculator.
- This test has seven questions, over five pages. **You should not answer all seven questions.**
  - The first two problems are two pages, representing topics M1 and M2. You should do both of them, and they are worth 30 points each.
  - The remaining five problems represent topics S1 through S5. You will be graded on your best three, with a few possible bonus points if you also do well on the other two.
  - Doing three secondary topics well is much better than doing five poorly.
  - If you perform well on a question on this test it will update your mastery scores. Achieving a 27/30 on a major topic or 9/10 on a secondary topic will count as getting a 2 on a mastery quiz.

Name:

Recitation Section:

	a	b	c
M1			
M2			
S1		S2	
S3		S4	
S5		$\sum$	

/90

Problem 1 (M1)**.** Compute the following using methods we have learned in class. Show enough work to justify your answers.

(a) Find an equation for the tangent line to $f(x) = e^{x^3+x}$ at the point $x = 1$.

Solution: We compute $f(1) = e^2$ and

$$f'(x) = e^{x^3+x}(3x^2+1)$$
$$f'(1) = e^2(4) = 4e^2$$

so the equation for the tangent line is

$$y - e^2 = 4e^2(x-1)$$

or

$$y = 4e^2(x-1) + e^2$$

or

$$y = 4e^2x - 3e^2.$$

(b) Compute $\displaystyle\int_2^e \frac{3}{x\ln(x)^3}\,dx$.

Solution: We can use integration by parts with $u = \ln(x)$, so $du = \frac{1}{x}\,dx$. Then we have

$$\int \frac{3}{x\ln(x)^3}\,dx = \int \frac{3}{u}^3\,du = \frac{-3}{2u^2} + C = \frac{-3}{2\ln(x)^2} + C$$
$$\int_2^3 \frac{3}{x\ln(x)^3}\,dx = \frac{-3}{2\ln(x)^2}\Big|_2^e = \frac{-3}{2} - \frac{-3}{2\ln(2)^2}.$$

Alternatively, we can change the bounds of integration. We know $u(2) = \ln(2)$ and $u(e) = 1$, so we get

$$\int_2^e \frac{3}{x\ln(x)^3}\,dx = \int_{\ln(2)}^1 \frac{3}{u^3}\,du = \frac{-3}{2u^2}\Big|_{\ln(2)}^1$$
$$= \frac{-3}{2} - \frac{-3}{2\ln(2)^2}.$$

(c) Compute $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$.

Solution: We can take $u = e^{2x}$ so $du = 2e^{2x}$. Then we have

$$\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx = \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du$$
$$= \frac{1}{2}\arcsin(u) + C. = \frac{1}{2}\arcsin(e^{2x}) + C.$$

Problem 2 (M2)**.** Compute the following integrals using methods we have learned in class. Show enough work to justify your answers.

(a) $\displaystyle\int \frac{x^3}{x^2-3x+2}\,dx =$

Solution: First we need to do a polynomial long division. We have

$$\begin{aligned} x^3 &= x(x^2 - 3x + 2) + 3x^2 - 2x \\ &= x(x^2 - 3x + 2) + 3(x^2 - 3x + 2) + 7x - 6 \\ \frac{x^3}{x^2 - 3x + 2} &= x + 3 + \frac{7x - 6}{x^2 - 3x + 2}. \end{aligned}$$

Then we need to do a partial fractions decomposition. We have

$$\begin{aligned} \frac{7x - 6}{x^2 - 3x + 2} &= \frac{A}{x - 2} + \frac{B}{x - 1} \\ 7x - 6 &= A(x - 1) + B(x - 2) \\ 1: \qquad 1 &= -B \\ 2: \qquad 8 &= A \\ \frac{7x - 6}{x^2 - 3x + 2} &= \frac{8}{x - 2} - \frac{1}{x - 1} \end{aligned}$$

so we have

$$\begin{aligned} \int \frac{x^3}{x^2 - 3x + 2}\,dx &= \int x + 3 + \frac{8}{x - 2} - \frac{1}{x - 1}\,dx \\ &= \frac{x^2}{2} + 3x + 8\ln|x - 2| - \ln|x - 1| + C. \end{aligned}$$

(b) $\displaystyle\int x^2 \sin(3x)\,dx =$

Solution:

$$\begin{aligned} \int x^2 \sin(3x)\,dx &= \frac{-1}{3}\cos(3x)x^2 - \int \frac{-1}{3}\cos(3x)\cdot 2x\,dx \\ &= \frac{-x^2}{3}\cos(3x) + \frac{2}{3}\int x\cos(3x)\,dx \\ &= \frac{-x^2}{3}\cos(3x) + \frac{2}{3}\left(\frac{1}{3}\sin(3x)x - \int \frac{1}{3}\sin(3x)\,dx\right) \\ &= \frac{-x^2}{3}\cos(3x) + \frac{2x}{9}\sin(x) + \frac{2}{27}\cos(3x) + C. \end{aligned}$$

(c) $\displaystyle\int \frac{x^3}{\sqrt{x^2 + 4}}\,dx =$

Solution: Here we should do a trigonometric substitution. We take $x = 2\tan(\theta)$, so we have $dx = 2\sec^2(\theta)\,d\theta$. Then

$$\begin{aligned} \int \frac{x^3}{\sqrt{x^2 + 4}}\,dx &= \int \frac{8\tan^3(\theta)}{\sqrt{4\tan^2(\theta) + 4}} 2\sec^2(\theta)\,d\theta \\ &= \int \frac{8\tan^3(\theta)}{\sqrt{4\sec^2(\theta)}} 2\sec^2(\theta)\,d\theta \\ &= \int \frac{8\tan^3(\theta)}{2\sec(\theta)} 2\sec^2(\theta)\,d\theta \\ &= \int 8\sec(\theta)\tan^3(\theta)\,d\theta. \end{aligned}$$

Now we need to use some trigonometric identities to do this integral. We know that things work if we have a single tangent function left, so we set $u = \sec(\theta)$, $du = \sec(\theta)\tan(\theta)\,d\theta$, and get

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+4}}\,dx &= \int 8\sec(\theta)\tan^3(\theta)\,d\theta \\ &= \int 8\sec(\theta)\tan(\theta)(\sec^2(\theta)-1)\,d\theta \\ &= 8\int u^2 - 1\,du = \frac{8}{3}u^3 - 8u + C = \frac{8}{3}\sec^3(\theta) - 8\sec(\theta) + C.\end{aligned}$$

Now we just need to figure out what $\sec(\theta)$ is. We know $\tan(\theta) = x/2$, so we get a triangle with opposite side x, adjacent side 2, and hypotenuse $\sqrt{x^2+4}$. Then we see that $\sec(\theta) = \frac{\sqrt{x^2+4}}{2}$, and thus the integral is

$$\int \frac{x^3}{\sqrt{x^2+4}}\,dx = \frac{(x^2+4)^{3/2}}{3} - 4\sqrt{x^2+4} + C.$$

Problem 3 (S1)**.** Let $f(x) = \sqrt[4]{x^7+5x^5+4x^3+6}$. Find $(f^{-1})'(2)$.

Solution: Plugging in numbers, we see that $f(0) = \sqrt[4]{6}$ but $f(1) = \sqrt[4]{16} = 2$, so $f^{-1}(2) = 1$. We compute

$$\begin{aligned}f'(x) &= \frac{1}{4}(x^7+5x^5+4x^3+6)^{-3/4}(7x^6+25x^4+12x^2) \\ f'(1) &= \frac{1}{4}(16)^{-3/4}(7+25+12) = \frac{44}{32} = \frac{11}{8}\end{aligned}$$

and thus by the Inverse Function Theorem,

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{11/8} = \frac{8}{11}.$$

Problem 4 (S2)**.** Find $\lim_{x\to+\infty} x^{1/\ln(1+2x)}$.

Solution: We can take

$$\begin{aligned}y &= x^{1/\ln(1+2x)} \\ \ln(y) &= \frac{1}{\ln(1+2x)}\ln(x) \\ \lim_{x\to+\infty}\ln(y) &= \lim_{x\to+\infty}\frac{\ln(x)^{\nearrow\infty}}{\ln(1+2x)_{\searrow\infty}} \\ &=^{\text{L'H}} \lim_{x\to+\infty}\frac{1/x}{2/(1+2x)} = \lim_{x\to+\infty}\frac{1+2x^{\nearrow\infty}}{2x_{\searrow\infty}} \\ &=^{\text{L'H}} \lim_{x\to+\infty}\frac{2}{2} = 1. \\ \lim_{x\to+\infty} y &= e.\end{aligned}$$

Problem 5 (S3)**.** Use the Trapezoid rule and four intervals to estimate $\int_{-1}^{3} x^3 - x^2\,dx$. Give an upper bound for the error on this approximation.

Solution:

$$\int_{-1}^{3} x^3 - x^2\,dx \approx \frac{-2+0}{2} + \frac{0+0}{2} + \frac{0+4}{2} + \frac{4+18}{2} = -1 + 2 + 11 = 12.$$

To find the error: If $f(x) = x^3 - x^2$ then $f''(x) = 6x - 2$, which is maximized at $f''(3) = 16$. So we can take $K = 16$ and have the formula

$$E_T \leq \frac{K(b-a)^3}{12n^2} = \frac{16(4)^3}{12 \cdot 4^2} = \frac{16}{3}.$$

Problem 6 (S4)**.** Compute $\displaystyle\int_{-\infty}^{-2} \frac{1}{x^3}\,dx$.

Solution:

$$\begin{aligned}
\int_{-\infty}^{-2} \frac{1}{x^3}\,dx &= \lim_{t\to-\infty} \int_{t}^{-2} \frac{1}{x^3}\,dx \\
&= \lim_{t\to-\infty} \frac{-1}{2} x^{-2}\Big|_{t}^{-2} \\
&= \lim_{t\to-\infty} \frac{-1}{8} - \frac{-1}{2t^2} = \frac{-1}{8}.
\end{aligned}$$

Problem 7 (S5)**.** Let $f(x) = \frac{1}{3}(x^2-2)^{3/2}$ Find the arc length of the graph of f for x between 2 and 3.

Solution: We compute $f'(x) = \frac{1}{2}(x^2-2)^{1/2} \cdot 2x = x\sqrt{x^2-2}$. So the arc length is

$$\begin{aligned}
L &= \int_2^3 \sqrt{1 + x^2(x^2-2)}\,dx = \int_2^3 \sqrt{1 - 2x^2 + x^4}\,dx \\
&= \int_2^3 x^2 - 1\,dx = \frac{x^3}{3} - x\Big|_2^3 = (9-3) - (8/3 - 2) = 8 - 8/3 = 16/3.
\end{aligned}$$