Math 1232 Spring 2023
Single-Variable Calculus 2 Section 12
Mastery Quiz 10
Due Tuesday, April 11

This week's mastery quiz has three topics. You should definitely submit all three of them.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Series Convergence
- Major Topic 4: Power and Taylor Series as functions
- Secondary Topic 8: Power Series

Name:

Recitation Section:

M3: Series Convergence

Analyze the convergence of the following three series. (Specify if they converge absolutely, converge conditionally, or diverge.)

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)+n}{n^2-1}$

Solution: You can't really use the limit comparison test here, at least not easily, because the numerator is a bit over-complicated. But you can use the usual comparison test. We know that $n \leq n + \ln(n)$ and $n^2 - 1 < n^2$, so

$$\frac{\ln(n)+n}{n^2-1} \geq \frac{n}{n^2} = \frac{1}{n}.$$

We know that $\sum \frac{1}{n}$ diverges by the p-series test, so our series diverges by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n-3}{n^2 4^n}$

Solution: We use the ratio test. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)^2+n+1-3 / (n+1)^2 4^{n+1}}{n^2+n-3 / n^2 4^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(n^2+2n-1)n^2 4^n}{(n+1)^2 4^{n+1}(n^2+n-3)} \right| \\ &= \lim_{n\to\infty} \left| \frac{n^4+2n^3-n^2}{4n^4+12n^3-20n-15} \right| = \frac{1}{4} < 1. \end{aligned}$$

So by the ratio test this converges absolutely..

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{2n+3}$

Solution: This is an alternating series. Since the terms $\frac{\sqrt{n}}{2m+3}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3}$. You can see this doesn't converge in a couple ways. The integral test isn't super plausible here. You can do a comparison test to $\frac{1}{\sqrt{n}}$: this is larger than $\frac{1}{3\sqrt{n}}$ for large n, and $\frac{1}{3\sqrt{n}}$ diverges. (note: this is *not* larger than $\frac{1}{\sqrt{n}}$!)

It may be easier to use the limit comparison test, though. We have

$$\lim_{n\to\infty} \frac{\sqrt{n}/2n+3}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{n}{2n+3} = 1/2.$$

Since the series $\sum \frac{1}{\sqrt{n}}$ diverges, by the limit comparison test, $\sum \frac{\sqrt{n}}{2n+3}$ diverges, and thus our series does not converge absolutely.

M4: Taylor Series

(a) Write a power series expression for $\frac{2x^2}{4x+1}$ centered at 0. What is the radius of convergence?

Solution: We know that

$$\begin{aligned}
\frac{1}{1-(-4x)} &= \sum_{n=0}^{\infty}(-4x)^n \\
\frac{2x^2}{1+4x} &= 2x^2\sum_{n=0}^{\infty}(-4)^n x^n \\
&= \sum_{n=0}^{\infty} 2\cdot(-4)^n x^{n+2} \\
\text{(or)} \quad &= \sum_{n=2}^{\infty} 2^{2n-3}(-1)^n x^n.
\end{aligned}$$

The radius of convergence is 1/4. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < -4x < 1$ or $-1/4 < x < 1/4$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{2n-1}(-1)^{n+1}x^{n+1}}{2^{2n-3}(-1)^n x^n}\right| = \lim_{n\to\infty} 4|x|$$

and thus it converges when $4|x| < 1$.

(b) If $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{n+1}{n!+1}x^n$, compute $\int_3^6 f(x)$.

Solution:

$$\begin{aligned}
\int f(x) &= \sum_{n=0}^{\infty}\frac{1}{n!+1}x^{n+1} + C \\
\int_3^6 f(x) &= \sum_{n=0}^{\infty}\frac{1}{n!+1}(6^{n+1} - 3^{n+1}).
\end{aligned}$$

(c) Write a power series expression for $\ln(1+x^2)$ centered at 1. What is the radius of convergence?

Solution: There are a few ways to approach this. One is to observe that $\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1+x^2}$. We know that

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$
$$\frac{2x}{1+x^2} = 2x\sum_{n=0}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$$

Then we can integrate both sides:

$$\ln(1+x^2) = \int \sum_{0}^{\infty} 2(-1)^n (x)^{2n+1}\,dx$$
$$= C + \sum_{0}^{\infty} 2(-1)^n \frac{x^{2n+2}}{2n+2}$$

and plugging 0 in on both sides tells us that $C = 0$. So our power series is

$$\ln(1+x^2) = \sum_{0}^{\infty} 2(-1)^n \frac{x^{2n+2}}{2n+2}.$$

Alternatively, you could recall that

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

so we can compute

$$\ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{2n}.$$

This looks different from our previous answer because it's been reindexed, but it is in fact the same answer. As an exercise, see if you can show they're the same!

S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=1}^{\infty} \frac{(2x-5)^n}{n^2}$.

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(2x-5)^{n+1}/(n+1)^2}{(2x-5)^n/n^2}\right| = \lim_{n\to\infty}\left|\frac{(2x-5)n^2}{(n+1)^2}\right|$$
$$= |2x-5|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = |2x-5|.$$

So we need $|2x-5| < 1$ or $-1 < 2x-5 < 1$, or $4 < 2x < 6$ or $2 < x < 3$. So the radius is $1/2$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty}\frac{(4-5)^n}{n^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2}$$

converges by alternating series test

$$\sum_{n=0}^{\infty}\frac{(6-5)^n}{n^2} = \sum_{n=0}^{\infty}\frac{1}{n^2}$$

converges by p-series test

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=1}^{\infty}\frac{n^2x^n}{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}$.

Solution: We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)^2x^{n+1}/1\cdot 3\cdot\dots\cdot(2n+1)}{n^2x^n/1\cdot 3\cdot\dots\cdot(2n-1)}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2x}{n^2(2n+1)}\right|$$
$$= |x|\lim_{n\to\infty}\frac{(n+1)^2}{n^2(2n+1)} = 0.$$

This is always less than 1, so the series always converges. The radius of convergence is ∞ and the interval of convergence is $(-\infty,\infty)$.