# Math 1232 Spring 2023
# Single-Variable Calculus 2 Section 12
# Mastery Quiz 11
# Due Tuesday, April 18

This week's mastery quiz has three topics. You should definitely submit M4. If you have a 4/4 on M3, or 2/2 on S8, you don't need to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

**Topics on This Quiz**

- Major Topic 3: Series Convergence
- Major Topic 4: Power and Taylor Series as functions
- Secondary Topic 8: Power Series

## Name:

## Recitation Section:

Name: Recitation Section:

# M3: Series Convergence

Analyze the convergence of the following three series. (Specify if they converge absolutely, converge conditionally, or diverge.)

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3+n}$

**Solution:** We use the Ratio test. We have

$$\lim_{n\to\infty} \left| \frac{(-2)^{n+1}/(n+1)^3+n+1}{(-2)^n/n^3+n} \right| = \lim_{n\to\infty} \frac{2(n^3+n)}{(n+1)^3+n+1}$$
$$= \lim_{n\to\infty} 2 > 1.$$

This limit is greater than 1, so by the ratio test this diverges.

Alternatively, we could note that

$$\lim_{n\to\infty} \frac{(-2)^n}{n^3+n} = \pm\infty,$$

so by the ratio test this diverges. But it's a little tricky to cleanly argue that this goes to infinity; we can't really use L'Hospital's rule without getting the negative sign out of there somehow.

(b) $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n}{n^2/5+3n}$

**Solution:** This clearly converges by the alternating series test, since $\lim_{n\to\infty} \frac{1}{n^2/5-3n} = 0$, but does it absolutely converge? The ratio test won't work; if we work it out we'll get a limit of 1. But we have

$$\sum_{n=4}^{\infty} \left| \frac{(-1)^n}{n^2/5+3n} \right| = \sum_{n=4}^{\infty} \frac{1}{n^2/5+3n},$$

so we can use the Limit Comparison Test to $\frac{1}{n^2}$. We compute

$$\lim_{n\to\infty} \frac{\frac{1}{n^2/5+3n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2/5+3n} = 1/5.$$

This is a nonzero real number, so since $\sum_{n=4}^{\infty} \frac{1}{n^2}$ converges, by the Limit Comparison Test, $\sum_{n=4}^{\infty} \frac{1}{n^2/5+3n}$ converges. Thus our original series converges absolutely. (And thus we don't actually need to check for whether the alternating series test applies.)

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2+1}$

**Solution:** We can work this out with the integral test. We have

$$\int_1^\infty xe^{-x^2+1}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2+1}\,dx \quad = \lim_{t\to\infty}\frac{-1}{2}e^{-x^2+1}\Big|_1^t = \lim_{t\to\infty}\frac{1}{2}e^2 - \frac{1}{2}e^{-t^2+1} = e^2/2 < \infty.$$

Since this integral converges, the series must also converge by the integral test. Since the series is all positive, it converges absolutely.

Alternatively, we could use the ratio test. We have

$$\begin{aligned}\lim_{n\to\infty}\frac{(n+1)e^{-n^2-2n}}{ne^{-n^2+1}} &= \lim_{n\to\infty}\frac{n+1}{n}\frac{e^{n^2-1}}{e^{n^2+2n}} \\ &= \lim_{n\to\infty}\frac{n+1}{n}\frac{1}{e^{2n+1}} = 0.\end{aligned}$$

Since $0 < 1$, this converges absolutely by the Ratio Test.

# M4: Taylor Series

(a) Let $f(x) = \cos^2(x)$. Use *the definition of a Taylor series* to find $T_4(x,\pi)$ for this function. (That is, find the terms up through the degree four term.)

**Solution:**

$$\begin{aligned}
f(x) &= \cos^2(x) & f(\pi) &= 1\\
f'(x) &= -2\cos(x)\sin(x) & f'(\pi) &= 0\\
f''(x) &= 2\sin^2(x) - 2\cos^2(x) & f''(\pi) &= -2\\
f'''(x) &= 4\sin(x)\cos(x) + 4\cos(x)\sin(x) & f'''(\pi) &= 0\\
f^{(4)} &= 8\cos^2(x) - 8\sin^2(x) & f^{(4)}(\pi) &= 8.
\end{aligned}$$

So we have

$$T_4(x,\pi) = 1 - (x-\pi)^2 + \frac{1}{3}(x-\pi)^4.$$

(b) In class we computed a Taylor series for $\sin(x)$ centered at zero. Use the degree-seven Taylor polynomial to approximate $\sin(3) \approx T_7(3,0)$. (You don't need to numerically simplify this.)

Using the Taylor series remainder, find an upper bound for the error in this approximation.

**Solution:** We know that

$$\begin{aligned}\sin(x) &= \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\\ T_7(x,0) &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\\ T_7(x,3) &= 3-\frac{27}{3!}+\frac{3^5}{5!}-\frac{3^7}{7!}=3-\frac{37}{6}+\frac{243}{120}-\frac{2187}{5040}\\ &= 3-\frac{9}{2}+\frac{81}{40}-\frac{243}{560}=\frac{51}{560}\approx 0.091.\end{aligned}$$

We know that $f^{n+1}(x)=\pm\cos(x)$ or $\pm\sin(x)$ so $|f^{n+1}(z)|\leq 1$, and thus

$$\begin{aligned}|R_n(x)| &= \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!}\\ |R_7(x)| &\leq \frac{x^{7+1}}{(7+1)!}\\ |R_7(3)| &\leq \frac{3^8}{8!}=\frac{729}{4480}\approx 0.16.\end{aligned}$$

It would also be okay to observe that the eighth term is zero, so we could actually compute

$$\begin{aligned}|R_n(x)| &= \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!}\\ |R_8(x)| &\leq \frac{x^{8+1}}{(8+1)!}\\ |R_8(3)| &\leq \frac{3^9}{9!}=\frac{243}{4480}\approx 0.054.\end{aligned}$$

(c) Using series we already know, write down a formula for the (infinite) Taylor series for $x^3e^{x^5/4}$, and then write down the first four non-zero terms of this series.

**Solution:**

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty}\frac{1}{n!}x^n\\ e^{x^5/4} &= \sum_{n=0}^{\infty}\frac{1}{n!}(x^5/4)^n=\sum_{n=0}\frac{1}{n!\cdot 4^n}x^{5n}\\ x^3e^{x^5/4} &= \sum_{n=0}\frac{1}{n!\cdot 4^n}x^{5n+3}\end{aligned}$$

The first four non-zero terms are

$$x^3 + \frac{1}{4}x^8 + \frac{1}{32}x^{13} + \frac{1}{6 \cdot 64}x^{18}.$$

(Note: this is *not* $T_3$ or $T_4$. It's $T_{18}$!)

# S8: Power Series

(a) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(5x-3)^n}{\sqrt{n}}$.

**Solution:** We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(5x-3)^{n+1}/\sqrt{n+1}}{(5x-3)^n/\sqrt{n}} \right| = \lim_{n\to\infty} \left| \frac{(5x-3)\sqrt{n}}{\sqrt{n+1}} \right|$$
$$= |5x-3| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = |5x-3|.$$

So we need $|5x-3| < 1$ or $-1 < 5x-3 < 1$, or $2 < 5x < 4$ or $2/5 < x < 4/5$. We need to have $x$ in the interval $(3/5 - 1/5, 3/5 + 1/5)$, so the radius is $1/5$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{(2-3)^n}{\sqrt{n}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by alternating series test

$$\sum_{n=0}^{\infty} \frac{(4-3)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by $p$-series test

Thus the interval of convergence is $[2/5, 4/5)$.

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n}{5^n}(x-3)^n$.

**Solution:** We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}/5^{n+1}}{(n)(x-3)^n/5^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{x-3}{5} \right|$$
$$= |x-3|/5 \lim_{n\to\infty} \frac{n+1}{n} = |x-3|/5.$$

So we need $|x-3|/5 < 1$ or $-5 < x-3 < 5$, or $-2 < x < 8$ or $3-5 < x < 3+5$. So the radius is 5.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{n}{5^n} 5^n = \sum_{n=0}^{\infty} n$$

diverges by divergence or $p$-series test

$$\sum_{n=0}^{\infty} \frac{n}{5^n} (-5)^n = \sum_{n=0}^{\infty} (-1)^n n$$

diverges by divergence test

Thus the interval is $(-2, 8)$.