Math 1232: Single-Variable Calculus 2
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Recitation 11

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Problem 1 (Bessel Function)**.** The Bessel function (of order 0) is critical to any physics done in cylindrical coordinates, and thus any physics that occurs on a cylinder. We saw it earlier as the solution to the differential equation $x^2y'' + xy' + x^2y = 0$, but it can also be given by the power series:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

What is the radius of convergence? What is the interval of convergence?

Solution: We use the ratio test. We have $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, so

$$\begin{aligned}
\lim \left| \frac{a_{n+1}}{a_n} \right| &= \lim \left| \frac{x^{2n+2}/2^{2n+2}((n+1)!)^2}{x^{2n}/2^{2n}(n!)^2} \right| \\
&= \lim \left| \frac{x^{2n+2}}{x^{2n}} \frac{2^{2n}}{2^{2n+2}} \frac{(n!)^2}{((n+1)!)^2} \right| \\
&= \lim \frac{|x|^2}{4(n+1)^2} = 0.
\end{aligned}$$

Thus the Bessel function of order 0 converges absolutely for all real numbers x. We say the radius of convergence is ∞ and the interval is all reals, or $(-\infty, +\infty)$.

Problem 2. What is the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{n^2(x-1)^n}{7^{n+2}}?$$

Solution: Using the ratio test, we have

$$\lim \left| \frac{(n+1)^2(x-1)^{n+1}/7^{n+3}}{n^2(x-1)^n/7^{n+2}} \right| = \lim \frac{|x-1|}{7} \frac{(n+1)^2}{n^2} = \frac{|x-1|}{7}.$$

So the series converges absolutely when $|x-1| < 7$, and thus on the interval $(-6, 8)$. For the full interval we need to test the endpoints, at $x = -6$ and $x = 8$.

When $x = -6$ we have

$$\sum \frac{n^2(-7)^n}{7^{n+2}} = \sum (-1)^n \frac{n^2}{49}.$$

This is an alternating series, but the terms tend towards infinity and so by the divergence test it diverges.

Similarly, when $x = 8$ we have

$$\sum \frac{n^2 7^n}{7^{n+2}} = \sum \frac{n^2}{49}.$$

The terms tend towards infinity, so the series diverges by the divergence test.

Thus the real interval of convergence is $(-6, 8)$.

Problem 3. Consider the function $f(x) = \frac{1}{1+x^6}$.

(a) Could you compute $\int \frac{1}{1+x^6}\,dx$? How?

(b) Does it help if I tell you that $1 + x^6 = (1+x^2)(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)$?

(c) Now write a power series for $f(x)$ centered at 0. What is the interval of convergence?

(d) Compute the integral of your power series. What is the interval of convergence there?

Solution:

(a) You'd have to do some sort of partial fractions thing.

(b) That helps a little, but I don't want to do that partial fractions thing and then I don't want to complete the square for those denominators.

(c)

$$\frac{1}{1+x^6} = \sum_{n=0}^{\infty} (-x^6)^n = \sum_{n=0}^{\infty} (-1)^n x^{6n}$$

This is a geometric series, so converges for $|-x^6| < 1$ and thus for $|x| < 1$.

(d)

$$\int \frac{1}{1+x^6}\,dx = \sum_{n=0}^{\infty} \int (-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{6n+1} + C.$$

This converges for $|x| < 1$. We can see that either by using the ratio test, or by knowing that integrating or differentiating can't change the radius of c convergence.

On the endpoints, we

Problem 4. We want to compute $\int_3^4 \frac{1}{1-(x-4)^3}\,dx$

(a) Find a power series for to compute $\frac{1}{1-(x-4)^3}$.

(b) Integrate the power series from 3 to 4. Does this converge?

(c) Sum the first five terms to estimate $\int_3^4 \frac{1}{1-(x-4)^3}\,dx$.

(d) Use an online integral calculator to find the integral. How close is your answer to the true answer?

Solution:

(a)

$$\frac{1}{1-(x-4)^3} = \sum_{n=0}^{\infty} (x-4)^{3n}$$

(b)

$$\begin{aligned} \int_3^4 \frac{1}{1-(x-4)^3}\,dx &= \sum_{n=0}^{\infty} \int (x-4)^{2n}\,dx \Big|_3^4 \\ &= \sum_{n=0}^{\infty} \frac{(x-4)^{3n+1}}{3n+1} \Big|_3^4 \\ &= 0 - \sum_{n=0}^{\infty} \frac{(-1)^{3n+1}}{3n+1} \end{aligned}$$

which converges by the Alternating Series Test.

(c)

$$\sum_{n=0}^{5} \frac{(-1)^{3n+1}}{3n+1} = 1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \frac{1}{13} - \frac{1}{16} = \frac{5877}{7280} \approx 0.80728.$$

(d) The true answer is about 0.835649 so this is prety decent.