Math 1232: Single-Variable Calculus 2
George Washington University   Spring 2023
Recitation 2

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Problem 1. (a) Compute $\log_3(6) + \log_3(9/2)$.

(b) Compute $\log_4(8) - \log_4(2)$.

(c) Rewrite the expression $\log_5(15) + \log_5(75) - \log_5(12)$ as an integer plus a logarithm.

(d) Solve $e^{5-3s} = 10$.

Solution:

(a) $\log_3(6) + \log_3(9/2) = \log_3(6 \cdot 9/2) = \log_3(27) = 3$.

(b) $\log_4(8) - \log_4(2) = \log_4(4) = 1$.

Alternatively $\log_4(8) - \log_4(2) = 1.5 - .5 = 14$.

(c)

$$\begin{aligned}\log_5(15) + \log_5(75) - \log_5(12) &= \log_5(15 \cdot 75/12) \\ &= \log_5\left(\frac{3}{4} \cdot 125\right) \quad = \log_5(125) + \log_5(3/4) = 3 + \log_5(3/4).\end{aligned}$$

You could also write this as $3 + \log_5(3) - \log_5(4)$ if you want.

(d) The problem here is that there's a variable in the exponent. To deal with difficult exponents, we take a logarithm. Then we see that $5 - 3x = \ln 10$ and so $x = \frac{5-\ln 10}{3}$.

Problem 2. A very important derivative is the derivative of $f(x) = \ln|x|$. There are two different approaches we can take here.

(a) If $x > 0$, can we write a formula for $f(x)$ without using a $|\cdot|$ sign? What is $f'(x)$ when $x > 0$?

(b) If $x < 0$, can we write a formula for $f(x)$ without using a $|\cdot|$ sign? What is $f'(x)$ when $x < 0$?

(c) What do those two answers tell you about $f'(x)$?

(d) There's another approach we can take. Think about the graph of $|x|$. What is $\frac{d}{dx}|x|$?

(e) Using the chain rule, what does this tell you about $f'(x)$?

(f) Does your answer from (e) match your answer from (c)?

Solution:

(a) If $x > 0$ then $f(x) = \ln(x)$, so $f'(x) = \frac{1}{x}$.

(b) If $x < -$ then $f(x) = \ln(-x)$. By the chain rule, $f'(x) = \frac{1}{-x} \cdot -1 = \frac{1}{x}$.

(c) Either way, we got $f'(x) = \frac{1}{x}$.

(d) From the graph we see that

$$\frac{d}{dx}|x| = \begin{cases} 1 & x > 0 \\ -1 x < 0 \end{cases}$$

which we can also write as $\frac{|x|}{x}$.

(e) Then

$$\frac{d}{dx} f(x) = \frac{1}{|x|} \cdot \frac{|x|}{x} = \frac{1}{x}.$$

(f) This gives us the same answer as before.

Problem 3. Compute the derivative of $(x + 1)^{\sqrt{x}}$.

Solution: We can't do this from the derivative rules we already have. But we can use logarithms!

$$\begin{aligned}
y &= (x+1)^{\sqrt{x}} \\
\ln|y| &= \ln\left|(x+1)^{\sqrt{x}}\right| = \sqrt{x}\ln|x+1| \\
\frac{y'}{y} &= \frac{1}{2\sqrt{x}}\ln|x+1| + \frac{\sqrt{x}}{|x+1|} \\
y' &= y\left(\frac{1}{2\sqrt{x}}\ln|x+1| + \frac{\sqrt{x}}{|x+1|}\right) \\
&= (x+1)^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\ln|x+1| + \frac{\sqrt{x}}{|x+1|}\right).
\end{aligned}$$

Problem 4 (Bonus)**.** Use logarithms to compute $\frac{d}{dx}\frac{x^3\sqrt{x^2-5}}{(x+4)^3}$.

Solution:

$$\begin{aligned}
y &= \frac{x^3\sqrt{x-5}}{(x+4)^3} \\
\ln|y| &= \ln\left|\frac{x^3\sqrt{x^2-5}}{(x+4)^3}\right| \\
&= 3\ln|x| + \frac{1}{2}\ln|x^2-5| - 3\ln|x+4| \\
\frac{y'}{y} &= \frac{3}{x} + \frac{1}{2}\frac{2x}{x^2-5} - \frac{3}{x+4} \\
y' &= y\left(\frac{3}{x} + \frac{1}{2}\frac{2x}{x^2-5} - \frac{3}{x+4}\right) \\
&= \frac{x^3\sqrt{x^2-5}}{(x+4)^3}\left(\frac{3}{x} + \frac{x}{x^2-5} - \frac{3}{x+4}\right).
\end{aligned}$$

If we want we can even simplify this to

$$y' = \frac{3x^2\sqrt{x^2-5}}{(x+4)^3} + \frac{x^4}{(x+4)^3\sqrt{x^2-5}} - \frac{3x^3\sqrt{x^2-5}}{(x+4)^4}$$

Problem 5. Compute the following integrals.

(a) $\int_e^{e^4} \frac{1}{x\sqrt{\ln x}}\,dx$.

(b) $\int e^x\cos(1+e^x)\,dx$.

(c) $\int \frac{\ln(x)}{x}\,dx$.

Solution:

(a) Take $u = \ln(x)$, and $du = \frac{dx}{x}$. $\ln(e) = 1$ and $\ln(e^4) = 4$. Then

$$\int_e^{e^4} \frac{1}{x\sqrt{\ln x}} dx = \int_1^4 \frac{1}{\sqrt{u}}\, du = 2\sqrt{u}\Big|_1^4 = 4 - 2 = 2.$$

(b) Take $u = 1 + e^x$ so $du = e^x dx$. Then

$$\int e^x \cos(1 + e^x)\, dx = \int \cos(u)\, du = \sin(u) + C = \sin(1 + e^x) + C.$$

(c) This one looks tricky, and you might have to mess around with it a bit to see, and try different things. But if we take $u = \ln(x)$ so that $du = \frac{1}{x}\, dx$, we see this is

$$\int u\, du = \frac{u^2}{2} + C = \frac{(\ln|x|)^2}{2} + C.$$

Problem 6 (Challenge)**.** Compute $\int \frac{dx}{1 + e^x}$.

Solution: This problem becomes much easier if we multiply the top and bottom by e^{-x}. Then we have $\int \frac{e^{-x}}{e^{-x} + 1}\, dx$. Set $u = e^{-x}$ so that $du = -e^{-x}\, dx$ and we have

$$\int \frac{e^{-x}}{e^{-x} + 1}\, dx = -\int \frac{du}{1 + u} = -\ln(1 + u) = -\ln(1 + e^{-x}).$$

Alternatively, we can take $u = e^x, du = e^x\, dx$, and have

$$\int \frac{dx}{1 + e^x} = \int \frac{du}{u(u + 1)}.$$

Again nonobviously, we write

$$\begin{aligned}\int \frac{du}{u(u+1)} = \int \frac{1 + u - u}{u(u + 1)}\, du &= \int \frac{1 + u}{u(u+1)}\, du - \int \frac{u}{u(u+1)}\, du \\ &= \int \frac{du}{u} - \int \frac{du}{u + 1} \\ &= \ln(u) - \ln(u + 1) = \ln(e^x) - \ln(e^x + 1).\end{aligned}$$

Using properties of logs, you can check that this is the same as the previous answer. Or, if you prefer, you can write $x - \ln(e^x + 1)$.