Math 1232: Single-Variable Calculus 2
George Washington University Spring 2023
Recitation 2

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Problem 1. (a) Compute $\sin(\arctan(5))$.

(b) Compute $\frac{d}{dx}\arccos(\sqrt{x})$

(c) Compute $\frac{d}{dx}\arctan(x+\sec(x))$

Solution:

(a) Our implicit triangle has side lengths of 5, 1, $\sqrt{26}$. So $\sin(\arctan(5)) = \frac{5}{\sqrt{26}}$.

(b)
$$\frac{d}{dx}\arccos(\sqrt{x}) = \frac{-1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}.$$

(c)
$$\frac{d}{dx}\arctan(x+\sec(x)) = \frac{1}{1+(x+\sec(x))^2} \cdot (1+\sec(x)\tan(x)).$$

Problem 2. Compute the following integrals:

(a) $\displaystyle\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$.

(b) $\displaystyle\int_0^1 \frac{e^{2x}}{1+e^{4x}}\,dx$.

(c)

Solution:

(a) Take $u = \arcsin(x)$, and $du = \frac{dx}{\sqrt{1-x^2}}$. Then

$$\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{1}{2}(\arcsin(x))^2 + C.$$

(b) Set $u = e^{2x}$ so $du = 2e^{2x}dx$. $g(0) = 1$ and $g(1) = e^2$. Then

$$\int_0^1 \frac{e^{2x}}{1+e^{4x}}\,dx = \int_1^{e^2} \frac{1}{2(1+u^2)}\,du = \frac{1}{2}\arctan(u)\Big|_1^{e^2} = \frac{1}{2}\left(\arctan(e^2) - \arctan(1)\right).$$

Problem 3. (a) In class, we saw that $\lim_{x\to+\infty} \frac{\ln(x)}{x} = 0$. What is $\lim_{x\to+\infty} \frac{\ln(x^2)}{x}$?

(b) Compute $\lim_{x\to+\infty} \frac{\ln(x^n)}{x}$ for $n > 0$.

(c) Compute $\lim_{x\to+\infty} \frac{\ln(x)}{x^\epsilon}$ for $\varepsilon > 0$.

(d) What do parts (a-c) tell you about the relationship between polynomials and $\ln(x)$?

Solution:

(a)

$$\lim_{x\to+\infty} \frac{\ln(x^2)}{x} = \lim_{x\to+\infty} \frac{2\ln(x)}{x} = 2\cdot 0 = 0$$

since we know this limit from class.

Alternatively

$$\lim_{x\to+\infty} \frac{\ln(x^2)^{\nearrow\infty}}{x_{\searrow\infty}} \overset{\text{L'H}}{=} \lim_{x\to+\infty} \frac{2x/x^2}{1} = \lim_{x\to+\infty} \frac{2^{\nearrow 2}}{x_{\searrow\infty}} = 0.$$

(b)

$$\lim_{x\to+\infty} \frac{\ln(x^n)}{x} = \lim_{x\to+\infty} \frac{n\ln(x)}{x} = n\cdot 0 = 0$$

$$\lim_{x\to+\infty} \frac{\ln(x)^{\nearrow\infty}}{x^{\varepsilon}{}_{\searrow\infty}} \overset{\text{L'H}}{=} \lim_{x\to+\infty} \frac{1/x}{\varepsilon x^{\varepsilon-1}}$$

$$= \lim_{x\to+\infty} \frac{1^{\nearrow 1}}{\varepsilon x^{\varepsilon}{}_{\searrow\infty}} = 0.$$

We see that $\ln(x)$ is *much much* smaller than any polynomial, when x is large. It doesn't matter how large a power we raise the inside of the logarithm to, or how small a power we raise the denominator to; in the limit, the logarithm will be infinitely smaller.

(a) In class we saw that $\lim_{x\to+\infty} \frac{e^x}{x} = +\infty$. Compute $\lim_{x\to+\infty} \frac{e^x}{x^2}$.

(b) Compute $\lim_{x\to+\infty} \frac{e^x}{x^n}$ for $n > 0$.

(c) What do parts (e-f) tell you about the relationship between e^x and polynomials?

Solution:

(a)

$$\lim_{x\to+\infty} \frac{e^x{}^{\nearrow\infty}}{x^2{}_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to+\infty} \frac{e^x{}^{\nearrow\infty}}{2x_{\searrow\infty}} =^{\text{L'H}} \lim_{x\to\infty} \frac{e^x{}^{\nearrow\infty}}{2_{\searrow 2}} = +\infty.$$

(b) To work this out formally we'd need a "proof by induction", but we can see what's happening.

$$\begin{aligned}\lim_{x\to+\infty} \frac{e^x{}^{\nearrow\infty}}{x^n{}_{\searrow\infty}} &=^{\text{L'H}} \frac{e^x{}^{\nearrow\infty}}{nx^{n-1}{}_{\searrow\infty}} \\ &=^{\text{L'H}} \frac{e^x{}^{\nearrow\infty}}{n(n-1)x^{n-2}{}_{\searrow\infty}} \\ &\vdots \quad \vdots \\ &=^{\text{L'H}} \frac{e^x{}^{\nearrow\infty}}{n(n-1)(n-2)\dots(3)(2)x_{\searrow\infty}} \\ &=^{\text{L'H}} \frac{e^x{}^{\nearrow\infty}}{n(n-1)(n-2)\dots(3)(2)_{\searrow n(n-1)\dots(3)(2)}} = +\infty.\end{aligned}$$

(c) This is the converse of the logarithm. e^x is much *bigger* than x^n for any positive n, when x is large; so e^x is asymptotically bigger than any polynomial.

Problem 4. (a) We want to compute $\lim_{x\to\pi/2} \sec(x) - \tan(x)$.

(b) Can we use L'Hospital's Rule on this as written? Can we change it to a form where L'Hospital's Rule works?

(c) What is the limit?

$$\begin{aligned}\lim_{x\to\pi/2} \sec(x) - \tan(x) &= \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)}\right) \\ &= \lim_{x\to\pi/2} \frac{1-\sin(x)^{\nearrow 0}}{\cos(x)_{\searrow 0}} \\ &=^{\text{L'H}} \lim_{x\to\pi/2} \frac{-\cos(x)^{\nearrow 0}}{-\sin(x)_{\searrow 1}} = \frac{0}{1} = 0.\end{aligned}$$

(d) Now let's compute $\lim_{x\to 0}\cot(2x)\sin(6x)$.

(e) Can we rewrite this so we can apply L'Hospital's Rule?

(f) What is the limit?

Solution:

(a) This is a $\infty - \infty$ limit.

(b) We can't use L'Hospital's Rule because this isn't a fraction. But we can write

$$\lim_{x\to\pi/2} \sec(x) - \tan(x) = \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)} \right)$$

(c)

$$\begin{aligned}
\lim_{x\to\pi/2} \sec(x) - \tan(x) &= \lim_{x\to\pi/2} \left(\frac{1}{\cos(x)} - \frac{\sin(x)}{\cos(x)} \right) \\
&= \lim_{x\to\pi/2} \frac{1 - \sin(x)^{\nearrow 0}}{\cos(x)_{\searrow 0}} \\
&=^{\text{L'H}} \lim_{x\to\pi/2} \frac{-\cos(x)^{\nearrow 0}}{-\sin(x)_{\searrow 1}} = \frac{0}{1} = 0.
\end{aligned}$$

(d) This is $0 \cdot \infty$.

(e)

$$\lim_{x\to 0} \cot(2x)\sin(6x) = \lim_{x\to 0} \frac{\sin(6x)\cos(2x)}{\sin(2x)}$$

(f)

$$\begin{aligned}
\lim_{x\to 0} \cot(2x)\sin(6x) &= \lim_{x\to 0} \frac{\sin(6x)\cos(2x)}{\sin(2x)} = 1 \cdot \lim_{x\to 0} \frac{\sin(6x)^{\nearrow 0}}{\sin(2x)_{\searrow 0}} \\
&= \lim_{x\to 0} \frac{6\cos(6x)^{\nearrow 6}}{2\cos(2x)_{\searrow 2}} = 3.
\end{aligned}$$

Problem 5. Let's compute $\lim_{x\to 0^+} x^{\frac{1}{\ln(x)-1}}$

(a) What indeterminate form is this?

(b) If $y = x^{\frac{1}{\ln(x)-1}}$, what is $\ln|y|$?

(c) Compute $\lim_{x\to 0^+} \ln|y|$.

(d) Compute $\lim_{x\to 0^+} x^{\frac{1}{\ln(x)-1}}$.

Solution:

(a) This is 0^0.

(b) $\ln(y) = \frac{1}{\ln(x)-1}\ln(x)$.

(c)

$$\lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} \frac{\ln(x)^{\nearrow \infty}}{\ln(x) - 1_{\searrow \infty}} =^{\text{L'H}} \lim_{x\to 0^+} \frac{1/x}{1/x} = 1$$

(d) $\lim_{x\to 0^+} y = e^1 = e$.