Math 1232 Spring 2023
Single-Variable Calculus 2 Section 12
Mastery Quiz 4
Due Tuesday, February 14

This week's mastery quiz has three topics. You should definitely submit M2. If you have a 2/2 on Blackboard in S2, you don't have to submit it. If you have a 4/4 on M1—meaning you've gotten it completely right twice—, then you don't have to submit it.

Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. Feel free to consult your notes, but please don't discuss the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Tuesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Calculus of Transcendental Functions
- Major Topic 2: Advanced Integration Techniques
- Secondary Topic 2: L'Hospital's Rule

Name:

Recitation Section:

M1: Calculus of Transcendental Functions

(a) $\displaystyle\int \frac{1}{4+x^2}\,dx =$

Solution: We can factor a 4 out to get $\frac{1}{4}\frac{1}{1x^2/4}$. Then we set $u = x/2$, and $du = 1/2\,dx$, and we have

$$\begin{aligned}\int \frac{1}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{4}\frac{2}{1+u^2}\,du \\ &= \int \frac{1}{2}\frac{1}{1+u^2}\,du \\ &= \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(x/2) + C.\end{aligned}$$

(b) (Note this is a definite integral)
$\displaystyle\int_0^2 \frac{e^{2x}}{e^{4x}+1}\,dx =$

Solution: We can take $u = e^{2x}$ so $du = 2e^{2x}\,dx$ and

$$\begin{aligned}\int_0^2 \frac{e^{2x}}{e^{4x}+1}\,dx &= \int_1^{e^4} \frac{1}{2}\frac{1}{u^2+1}\,du = \frac{1}{2}\arctan(u)\Big|_1^{e^4} \\ &= \frac{1}{2}\arctan(e^4) - \frac{1}{2}\arctan(1) = \frac{1}{2}\arctan(e^4) - \frac{\pi}{8}.\end{aligned}$$

(c) Compute $\displaystyle\frac{d}{dx}\left(\sqrt{x+1}\right)^x$

Solution:

$$\begin{aligned}y &= \sqrt{x+1}^x \\ ln|y| &= x\ln(\sqrt{x+1}) = \frac{1}{2}x\ln(x+1) \\ y'/y &= \frac{1}{2}\left(\ln(x+1) + \frac{x}{x+1}\right) \\ y' &= \frac{1}{2}\sqrt{x+1}^x\left(\ln(x+1) + \frac{x}{x+1}\right)\end{aligned}$$

M2: Advanced Integration Techniques

(a) $\displaystyle\int \sin(2x)\cos(3x)\,dx =$

(Please do *not* use any product-of-trig-function identities we haven't discussed in class.)

Solution:

$$\begin{aligned}
\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) - \int \frac{2}{3}\cos(2x)\sin(3x)\,dx \\
\int \cos(2x)\sin(3x)\,dx &= -\frac{1}{3}\cos(2x)\cos(3x) - \int \frac{2}{3}\sin(2x)\cos(3x) \\
\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}\int \sin(2x)\cos(3x)\,dx \\
\frac{5}{9}\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x)\big(+C\big) \\
\int \sin(2x)\cos(3x)\,dx &= \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.
\end{aligned}$$

(b) $\int \cos^5(2x)\,dx =$

Solution:

$$\begin{aligned}
\int \cos^5(2x)\,dx &= \int \cos(2x)(1-\sin^2(2x))^2\,dx \\
&= \int \cos(2x) - 2\sin^2(2x)\cos(2x) + \sin^4(2x)\cos(2x)\,dx \\
&\frac{1}{2}\sin(2x) - \frac{1}{3}\sin^3(2x) + \frac{1}{10}\sin^5(2x) + C.
\end{aligned}$$

(If you want you can explicitly do the substitution $u = \sin(2x), du = 2\cos(2x)\,dx$, but you don't have to write it out explicitly.)

(c) $\int \sec^4(3t)\,dt =$

Solution: We're going to take $u = \tan(3t)$ so that $du = 3\sec^2 3t\,dt$. Then

$$\begin{aligned}
\int \sec^4(3t)\,dt &= \int \sec^2(3t)(1+\tan^2(3t))\,dt \\
&= \int \frac{1}{3}(1+u^2)\,du \\
&= \frac{u}{3} + \frac{u^3}{9} + C \\
&= \frac{1}{9}\big(3\tan(3t) + \tan^3(3t)\big) + C.
\end{aligned}$$

(d) $\int \frac{dx}{x^2\sqrt{4-x^2}} =$

Solution: We can take $x = 2\sin(\theta)$, so that $dx = 2\cos(\theta)\,d\theta$. Then we get

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{4-x^2}} &= \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\sqrt{4-4\sin^2(\theta)}} \\
&= \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\sqrt{4\cos^2(\theta)}} \\
&= \int \frac{2\cos(\theta)\,d\theta}{4\sin^2(\theta)\cdot 2\cos(\theta)} \\
&= \int \frac{d\theta}{4\sin^2(\theta)} = \int \frac{1}{4}\csc^2(\theta)\,d\theta \\
&= -\frac{1}{4}\cot(\theta) + C.
\end{aligned}$$

Now we need to substitute our x back in. We know that $\sin(\theta) = x/2$, so we can construct a right triangle with opposite side x, hypotenuse 2, and thus adjacent side $\sqrt{4-x^2}$. (Note that this is the term that showed up in the original integral!)

Then $\cot(\theta) = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{4-x^2}}{x}$, and so we have

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = -\frac{1}{4}\cot(\theta) + C = -\frac{\sqrt{4-x^2}}{4x} + C.$$

S2: L'Hospital's rule

(a) $\displaystyle \lim_{x\to 0} \left(\frac{e^x+1}{2}\right)^{1/x} =$

Solution:

$$\begin{aligned}
\ln y &= \frac{1}{x}\ln\left(\frac{e^x+1}{2}\right) \\
\lim_{x\to 0} \ln y &= \lim_{x\to 0} \frac{\ln\left(\frac{e^x+1}{2}\right)^{\nearrow 0}}{x_{\searrow 0}} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{2}{e^x+1}\cdot\frac{e^x}{2} = \lim_{x\to 0}\frac{e^x}{e^x+1} = 1/2 \\
\lim_{x\to 0} y &= e^{1/2}.
\end{aligned}$$

(b) $\displaystyle \lim_{x\to+\infty} \frac{\arctan(x)}{\arctan(x)+1} =$

Solution: $\lim_{x\to+\infty} \arctan(x) = \pi/2$, so this limit is $\frac{\pi/2}{\pi/2+1} \approx .611$.

Note: you cannot use L'Hospital's rule here! If you tried, you would get

$$\lim_{x\to+\infty} \frac{1/(x^2+1)}{1/(x^2+1)} = \lim_{x\to+\infty} 1 = 1$$

but that is not in fact the limit.

(c) $\displaystyle\lim_{x\to 1} \frac{\ln(x)}{\arcsin(2x-2)} =$

Solution: The top and bottom both approach 0, so we can use L'Hospital's Rule:

$$\lim_{x\to 1} \frac{\ln(x)^{\nearrow 0}}{\arcsin(x-1)_{\searrow 0}} \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{1/x}{\frac{2}{\sqrt{1-(2x-2)^2}}}$$

$$= \lim_{x\to 1} \frac{\sqrt{1-(x-1)^2}}{2x} = \frac{1}{2}.$$